Optimal Control of Martingales in a Radially Symmetric Environment

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Problem Statement

Minimise

$$\mathbb{E}\left[\int_0^{\tau_D} f(X_s)\,\mathrm{d}s + g(X_{\tau_D})\right]$$

over all continuous martingales X with unit quadratic variation, defined on some bounded domain

$$D \subset \mathbb{R}^d.$$

Motivation

The Martingale Optimal Transport (MOT) problem is to find

$$\mathcal{V}(\mu_0, \mu_1) = \inf_{\pi \in \Pi_M(\mu_0, \mu_1)} \int_{\mathbb{R}^d} c(x, y) \, \mathrm{d}\pi(x, y).$$

- $d = 1$:
  - MOT is well-understood
  - Any martingale is a time change of Brownian motion
- $d \geq 2$:
  - Structure of martingale transports is more complicated
  - Some recent progress has been made - e.g. [Lim, 2014, Ghoussoub et al., 2019]
  - We attempt to understand solutions to martingale control problems

Problem Formulation

Problem Formulation

We seek the value function

$$v(x) := \inf_{\mathbb{P}\in\mathcal{P}_x} \mathbb{E}^{\mathbb{P}}\left[\int_0^\tau f(X_s)\,\mathrm{d}s + g(X_\tau)\right],$$

where $\mathcal{P}_x$ is the set of probability measures on $\Omega \times \mathcal{B}(\mathbb{R}_+, U)$

$$\begin{cases} \Omega = C(\mathbb{R}_+, D) & \text{- path space} \\ U = \{\sigma \in \mathbb{R}^{d,d} \colon \ \mathrm{Tr}(\sigma\sigma^\top) = 1\} & \text{- control set} \end{cases}$$

under which

$$t \mapsto \phi(X_t) - \phi(X_0) - \frac{1}{2}\int_0^t \mathrm{Tr}(D^2\phi(X_s)\nu_s\nu_s^\top)\,\mathrm{d}s$$

is a martingale for any $\phi \in C^2(D)$ with the restriction that $\mathbb{P}(X_0 = x) = 1$ for all $\mathbb{P} \in \mathcal{P}$.

We will refer to this as the weak formulation.

Assumptions

$$v(x) := \inf_{\mathbb{P} \in \mathcal{P}_x} \mathbb{E}^{\mathbb{P}} \left[\int_0^\tau f(X_s)\,\mathrm{d}s + g(X_\tau) \right],$$

1. $D = B_R(0) \subset \mathbb{R}^d$
2. f radially symmetric; i.e. $f(x) = \tilde{f}(|x|)$
3. g constant
4. f continuous
5. $\tilde{f}'(r+)$ exists for all $r \geq 0$ with $\lim_{r\to 0} r\tilde{f}'(r) = 0$

Strong Formulation

Under the above conditions, the problem is equivalent to the following strong formulation [El Karoui and Tan, 2013].

Fix a probability space on which a d-dimensional Brownian motion B is defined, with natural filtration $\mathbb{F}$.

Find

$$v^S(x) := \inf_{\sigma \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X_s^\sigma)\, \mathrm{d}s + g(X_\tau^\sigma) \right],$$

where $\mathcal{U}$ is the set of $\mathbb{F}$-progressively measurable U-valued processes and

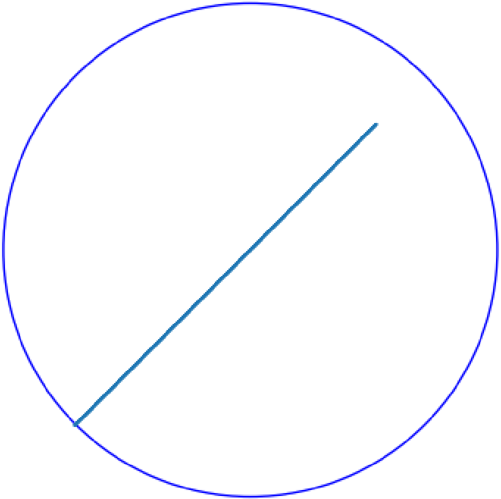
$$\mathrm{d}X_t = \sigma_t \, \mathrm{d}B_t \quad \text{for all} \quad \sigma \in \mathcal{U},$$

$$U = \{\sigma \in \mathbb{R}^{d,d} \colon \operatorname{Tr}(\sigma\sigma^\top) = 1\}.$$

Optimal Behaviour

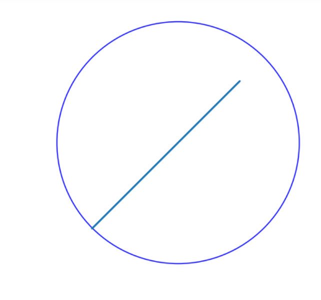
Radial Motion

Optimal behaviour for $\tilde{f}$ increasing

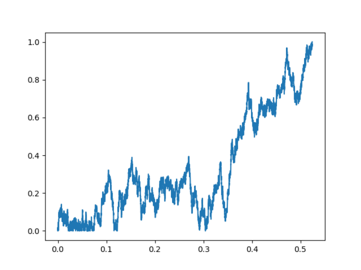


Radial Motion

- Control: $$\sigma_t = \frac{1}{|X_t|}[X_t; 0; \dots ; 0]$$
- Radius process: $$\mathrm{d}R_t = \mathrm{d}W_t$$
- Generator: $$\mathcal{L}u(r) = \frac{1}{2}u''(r)$$



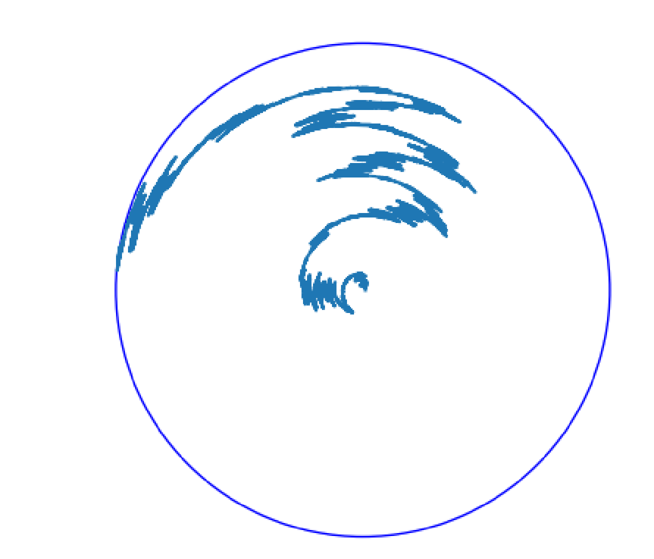
Sample path of X_t



Sample path of R_t

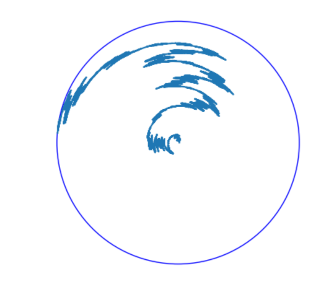
Tangential Motion

Optimal behaviour for $\tilde{f}$ decreasing

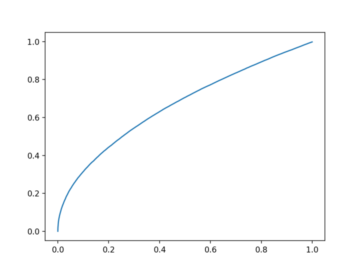


Tangential Motion

- Control: $$\sigma_t = \frac{1}{|X_t|}[X_t^{\perp}; 0; \ldots ; 0]$$
- Radius process: $$\mathrm{d}R_t = \frac{1}{2R_t}\,\mathrm{d}t$$
- Generator: $$\mathcal{L}u(r) = \frac{1}{2r}u'(r)$$

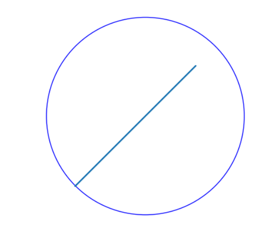


Sample path of X_t

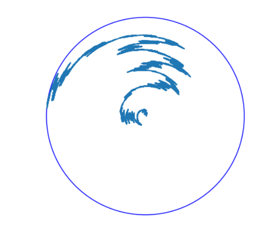


Sample path of R_t

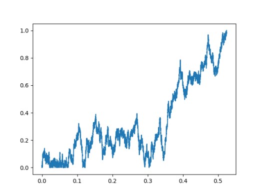
Two optimal behaviour regimes



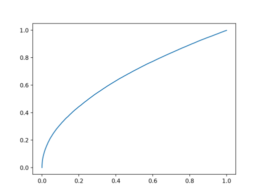
(a) Sample path of radial motion



(b) Sample path of tangential motion



(c) Sample path of radius process for (a)



(d) Sample path of radius process for (b)

Construction of Solution

Claim that the optimal strategy is to switch between radial and tangential motion.

Then $v(x) = \tilde{v}(|x|)$, where $\tilde{v}$ solves

$$\begin{cases} \min\left\{\frac{1}{2}\tilde{v}''(r), \frac{1}{2r}\tilde{v}'(r)\right\} = -\tilde{f}(r), \quad r \in (0, r), \\ \tilde{v}(R) = g. \end{cases}$$

So to minimise

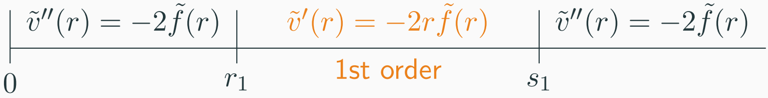
$$\tilde{v}(r) = g - \int_r^R \tilde{v}'(s)\,\mathrm{d}s,$$

we seek to maximise $\tilde{v}'(r)$.

Construction of Solution

Maximise $\tilde{v}'$, where $\tilde{v}$ solves

$$\min\left\{\frac{1}{2}\tilde{v}''(r), \frac{1}{2r}\tilde{v}'(r)\right\} = -\tilde{f}(r).$$



Switching point is determined by

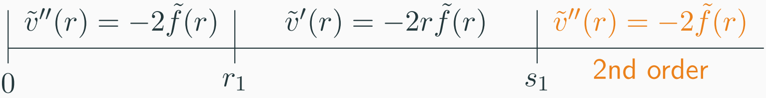
$$r_1 = \inf\{s > s_0 \colon \tilde{v}'(s) < -2s\tilde{f}(s)\}.$$

By continuity of f, we have smooth fit at r_1, even though the local time is zero.

Construction of Solution

Maximise $\tilde{v}'$, where $\tilde{v}$ solves

$$\min\left\{\frac{1}{2}\tilde{v}''(r), \frac{1}{2r}\tilde{v}'(r)\right\} = -\tilde{f}(r).$$



We need to enforce smooth fit at s_1, and we need a 2nd order condition to determine the switching point:

$$s_1 = \inf\{r > s_0 \colon \tilde{v}''(r) < -2\tilde{f}(r)\}.$$

Value Function

Continue in this way to construct a sequence of switching points

$$r_0 < s_0 < \ldots < r_i < s_i < \ldots .$$

Solve the ODEs by imposing smooth fit at points s_i and continuous fit at points r_i, s_i.

We arrive at the following candidate value function:

$$V(x) = \begin{cases} -2\int_{s_{i-1}}^{|x|}\int_{s_{i-1}}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s - 2\,|x|\,s_{i-1}\tilde{f}(s_{i-1}) + C_i^u, & |x| \in [s_{i-1}, r_i], \\ -2\int_{r_i}^{|x|} s\tilde{f}(s)\,\mathrm{d}s + C_i^w, & |x| \in [r_i, s_i]. \end{cases}$$

Proof of Optimality

We use the theory of viscosity solutions to show optimality:

1. The value function v is continuous and M-convex
2. v satisfies a dynamic programming principle
3. v is a viscosity solution to

$$\begin{cases} \inf_{\sigma \in U} \operatorname{Tr}(D^2 v \sigma \sigma^\top) = -f & \text{in} \quad D \\ v = g & \text{on} \quad \partial D \end{cases} \tag{HJB}$$

4. The candidate function V solves (HJB)
5. Comparison holds for (HJB)

Hence $v = V$.

Relaxing Assumptions

Exploding Cost at Origin

We now relax the assumptions:

$$v(x) := \inf_{\mathbb{P}\in\mathcal{P}_x} \mathbb{E}^{\mathbb{P}}\left[\int_0^\tau f(X_s)\,\mathrm{d}s + g(X_\tau)\right],$$

1. $D = B_R(0)$
2. f radially symmetric; i.e. $f(x) = \tilde{f}(|x|)$
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5. $\tilde{f}'(r+)$ exists for all $r \geq 0$ with $\lim_{r\to 0} r\tilde{f}'(r) = 0$

Exploding Cost at Origin

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$$v(x) := \inf_{\mathbb{P}\in\mathcal{P}_x} \mathbb{E}^{\mathbb{P}}\left[\int_0^\tau f(X_s)\,\mathrm{d}s + g(X_\tau)\right],$$

1. $D = B_R(0)$
2. f radially symmetric; i.e. $f(x) = \tilde{f}(|x|)$
3. g constant
4. f continuous $D \setminus \{0\}$ and $\lim_{r\to 0} r^\beta \tilde{f}(r) = \alpha$, for some α, β
5. $\tilde{f}'(r+)$ exists for all $r \geq 0$

Exploding Cost at Origin

Growth condition:

$$\lim_{r \to 0} r^{\beta} \tilde{f}(r) = \alpha \leq 0.$$

Radial motion is optimal near the origin and:

- For $\beta < 1, \quad v = v^S = V > -\infty$
- For $\beta \geq 1, \quad v = v^S \equiv -\infty$

Prove using Green's function for $\mathrm{d}R_t = \mathrm{d}W_t$, $0 < r < \eta$:

$$\mathbb{E}^r \left[\int_0^{\tau_\eta} \tilde{f}(R_s) \, \mathrm{d}s \right] \sim \int_{-\eta}^{\eta} \tilde{f}(\xi) \, \mathrm{d}\xi + \int_{-\eta}^{\eta} \xi \tilde{f}(\xi) \, \mathrm{d}x.$$

Exploding Cost at Origin

Growth condition:

$$\lim_{r \to 0} r^{\beta} \tilde{f}(r) = \alpha > 0.$$

Conjecture: We can construct a martingale X such that

- $X_0 = 0$
- $R_t = |X_t|$ satisfies

$$\mathrm{d}R_t = \frac{1}{2R_t}\,\mathrm{d}t$$

but X will not be adapted to a Brownian filtration.

Hence we expect the strong and weak control problems to differ.

Exploding Cost at Origin

Growth condition:

$$\lim_{r \to 0} r^{\beta} \tilde{f}(r) = \alpha > 0.$$

Conjecture:

- For $\beta < 1$, $\quad v = v^S < \infty$;
- For $\beta \in [1, 2)$, $\quad v < \infty$ but $v^S(0) = \infty$;
- For $\beta \geq 2$, $\quad v(0) = v^S(0) = \infty$, $v(x), v^S(x) < \infty$, $x \neq 0$.

Idea is that if $\mathrm{d}R_t = \frac{1}{2R_t}\,\mathrm{d}t$, then

$$\mathbb{E}^r \left[\int_0^{\tau_\eta} \tilde{f}(R_s)\,\mathrm{d}s \right] = 2 \int_0^{\eta} \xi \tilde{f}(\xi)\,\mathrm{d}\xi$$

Discontinuous Cost

Fix $R_0 \in (0, R)$ and define

$$f(x) = \begin{cases} 0, & |x| \leq R_0 \\ -1, & |x| \in (R_0, R). \end{cases}$$

Tangential motion is optimal and the value function is

$$v(x) = \begin{cases} R_0^2 - R^2, & |x| \leq R_0 \\ |x|^2 - R^2, & |x| \in (R_0, R). \end{cases}$$

Prove using the Itô-Tanaka formula.

v is a viscosity solution to (HJB) as in [Cattiaux et al., 2008].

However, there is no uniqueness theory for viscosity solutions with discontinuous data.

Summary

- Optimal behaviour is either radial or tangential motion
- Switch between 1st & 2nd order behaviour with smooth fit
- Identified growth condition at origin to give finite value
- Will investigate constructing weak solution at origin

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