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Constrained Optimal Stopping Problems

Thesis Formulation Report

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Abstract

Constrained optimal stopping problems have been approached in recent literature by transforming the problem into either an unconstrained problem, using Lagrange multipliers, or into a stochastic optimal control problem, by introducing an auxiliary process. We focus on the second approach here and explore a wider class of stochastic optimal control problems and their corresponding Hamilton-Jacobi-Bellman PDEs. We also propose a relationship between constrained optimal stopping problems, Monge-Ampère equations, martingale optimal transport and Skorokhod embedding.

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Introduction

In this report we present some preliminary work on constrained optimal stopping and related problems, and we outline some ideas which we intend to explore further in the course of a PhD thesis.

An optimal stopping problem is to find the stopping time which maximises the expectation of a certain quantity calculated from the path of a stochastic process. Typically, such a problem is of the following form, as described in Chapter III.6 of [26]:

$$v(x) := \sup_{\tau} \mathbb{E}^x \left[M(X_\tau) + \int_0^\tau L(X_t)\,\mathrm{d}t + \sup_{0\le t<\tau} K(X_t) \right],$$

for some functions M, L, K and a stochastic process $(X_t)_{t\ge0}$, where the supremum is taken over all stopping times.

This type of problem has been well-studied and there are many techniques for its solution, several of which are detailed in [26]. It is also worth noting that these problems have a financial application, explained in Chapter VII of [26], as methods of pricing options and determining the optimal exercise time for options.

Classical optimal stopping problems can be solved on either an infinite or a finite time horizon, as described in [26]. However, adding a constraint on the *expected value* of the stopping time poses an additional challenge. We will call a problem with such a constraint a *constrained optimal stopping problem*, and it is this type of problem that we begin this report by studying.

The first example of a constrained optimal stopping problem which we are aware of in the literature is the 1982 paper of Kennedy [20]. Since 2015, several new papers have appeared on this type of problem, including [1,19,21,25]. A similar type of constrained problem is considered by Bayraktar and Miller in [3], where the distribution of the stopping time is constrained to be a given measure. We note that this problem may be related to the other problems and methods which we are considering, and could be of interest to explore further in the future, but here we concentrate on problems with a constraint on the expectation of the stopping time.

The study of constrained optimal stopping problems is financially motivated by mean-variance investment strategies, as explained in the paper of Pedersen and Peskir [25]. In that paper, the model for a stock price $(X_t)_{t\ge0}$ is taken to be a geometric Brownian motion: $\mathrm{d}X_t = \sigma X_t\,\mathrm{d}B_t + \mu X_t\,\mathrm{d}t$, where B is a standard Brownian motion. A mean-variance investment strategy is a simple strategy for maximising reward while minimising risk, using the expected value of X as a proxy for reward and the variance of X as a proxy for risk. This can be formalised as a constrained optimal stopping problem of finding either

$$\sup_{\tau} \mathbb{E}^x[X_\tau] \quad \text{subject to} \quad \mathrm{Var}^x(X_\tau) \le \alpha \qquad \text{or} \qquad \sup_{\tau} \mathrm{Var}^x[X_\tau] \quad \text{subject to} \quad \mathbb{E}^x(X_\tau) \le \beta,$$

for some $\alpha, \beta > 0$. The first of these formulations can be related to the optimal stopping problem with expectation constraint described above.

There are two main strategies which have been successful for solving constrained optimal stopping problems. Kennedy used Lagrange multipliers in [20] to transform the constrained problem into an unconstrained optimal stopping problem, and Pedersen and Peskir used a similar method in [25], allowing them to solve the problem as a free boundary problem. Miller uses a second method in [21], with Ankirchner, Klein and Kruse in [1] and Karnam, Ma and Zhang in [19] taking essentially the same approach. In these papers, the authors introduce an auxiliary process to turn the constrained optimal stopping problem into a two-process *stochastic optimal control problem.*

A stochastic optimal control problem is defined in Section 2.1 of [33] as follows. Let X be a stochastic process defined by

$$\mathrm{d}X_t = \mu(t, X_t, \alpha_t)\,\mathrm{d}t + \sigma(t, X_t, \alpha_t)\,\mathrm{d}B_t,$$

for some stochastic *control process* α, which is chosen from a set of admissible controls U. Then the problem is to find the *value function*

$$v(t, x) := \sup_{\alpha \in U} J(t, x, \alpha),$$

where J is the expected value of a reward function depending on the path of the controlled stochastic process X.

As stated in [33], the value function v solves a dynamic programming or Hamilton-Jacobi-Bellman equation, which is a partial differential equation. Verifying that a function satisfies the appropriate Hamilton-Jacobi-Bellman equation is a common method of finding a solution to the corresponding stochastic control problem, and this approach is used in the papers [1, 21]. The appropriate notion of solution to these PDEs to use here is the viscosity solution, which we define in Section 1.5, as described in [15].

In [21], Miller observes that the Hamilton-Jacobi-Bellman equation arising from the constrained optimal stopping problem considered there is a Monge-Ampère type equation. The Monge-Ampère equation is a non-linear PDE, a specific form of which is a key equation in the theory of optimal transport, as detailed in [34] and [10]. Spiliotis and Karatzas have both proved stochastic representation results for parabolic Monge-Ampère type equations in terms of stochastic optimal control problems, in [32] and [18], respectively. However, we do not know of any such result for the form of the Monge-Ampère equation which appears in optimal transport, and it would be of interest to derive a stochastic representation result for more general Monge-Ampère equations.

Several recent papers have considered the problem of imposing an additional martingale constraint on the classical Monge-Kantorovich optimal transport problem, and we begin to explore *martingale optimal transport* in this report. Suppose we wish to transport mass distributed according to a measure μ to be distributed according to a second measure λ, while minimising some cost function. Then the martingale version of this optimal transport problem imposes the additional condition that we must have $\mathbb{E}[Y|X] = X$, when $\mathrm{Law}(X) = \mu$ and $\mathrm{Law}(Y) = \lambda$, as described in [6].

A financial motivation for martingale optimal transport comes from model-free or robust pricing of derivatives, as illustrated, for example, by Hobson and Klimmek in [16]. Traditionally, financial derivatives have been priced by first specifying a model which they follow, then calibrating this model by using observed prices to estimate model parameters, as described in [29]. However, choosing the right model is extremely difficult, and almost no model can account for extreme changes in the market. Therefore, recent research in mathematical finance has explore model-free pricing of derivatives, in which no model is specified, but bounds on the price can be obtained based only on observations. In [16], we see that this method of pricing can lead to a martingale optimal transport problem.

We suggest that there may be a link between the financial setup here and the mean-variance strategies discussed above, and that we might therefore expect a link between constrained optimal stopping and martingale optimal transport. Furthermore, in [6], Beiglböck and Juillet showed a monotonicity principle for martingale optimal transport, analogous to the well-known monotonicity result in the classical case. This monotonicity principle has some similarity to the properties of optimal paths which we will observe for constrained optimal stopping problems.

Furthermore, in [5], Beiglböck, Henry-Labordère and Touzi make a connection between the monotonicity property of martingale optimal transport and a *Skorokhod embedding problem*. Skorokhod embedding problems are the final set of problems which we consider in this report. As defined in [23], the embedding problem is to find a stopping time τ such that the law of B_τ is a given measure. Since this problem was first introduced in 1961 by Skorokhod [30], many different solutions have been posed, including those presented in [2,11,28]. In [4], the authors use techniques from the theory of optimal transport to solve optimal Skorokhod embedding problems, which have the additional property that their solution must minimise a given functional of the Brownian motion.

We believe that there exists an underlying connection between all of the classes of problems which we consider in this report. We now outline the structure of the report.

In Chapter 1, we introduce a constrained optimal stopping problem, which we reformulate as a stochastic optimal control problem. We solve this problem explicitly in the simplest case and introduce the machinery of the dynamic programming principle and viscosity solutions, which will allow us to approach more general problems in the future.

In Chapter 2, we note that the PDE arising from the stochastic control problem of Chapter 1 resembles a Monge-Ampère equation, which leads us to investigate stochastic representation results for Monge-Ampère equations. We provide one preliminary result in this report and conjecture a more general set of results. We also point to a possible connection to the theory of optimal transport, which we wish to investigate.

In Chapter 3, we introduce martingale optimal transport problems, compare them to classical optimal transport and give some results on these problems from the literature. In addition, we define Skorokhod embedding problems and describe some of their basic features. We conclude by proposing a connection which would tie martingale optimal transport and Skorokhod embedding back to the Monge-Ampère equation, stochastic optimal control and constrained optimal stopping problems.

The aim of this report is to introduce several related problems and present preliminary results, while outlining opportunities for future avenues of research.

Chapter 1

Constrained Optimal Stopping

The following optimal stopping problem with a constraint on the expected value of the stopping time has been studied in recent papers including [1, 19, 21, 25]:

$$\begin{aligned} \text{Find} \quad & \sup_{\tau} \mathbb{E}^x[f(B_\tau)] \\ \text{subject to} \quad & \mathbb{E}^x[\tau] \leq K, \end{aligned} \tag{1.0.1}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion and the supremum is taken over all stopping times.

In this problem, we have a reward function $f : \mathbb{R} \to \mathbb{R}$, which depends only on the value of the Brownian motion at the stopping time, and our aim is to maximise the expected reward.

Successful approaches to this problem have followed one of two main strategies. First, the approach of Pedersen and Peskir in [25] is to use Lagrange multipliers to transform the constrained problem into an unconstrained free boundary problem, which can then be solved. The second approach, which leads to the same solution, is to introduce an auxiliary process in order to reformulate the problem as a stochastic optimal control problem. This approach is taken, for example, by Miller in [21], where PDE methods are then used to arrive at the solution.

We focus now on the approach of Miller and, in this chapter of the report, we explain and justify the approach that we will take to this type of problem, and we present some simple examples, which we plan to build on in the future.

1.1 Stochastic Optimal Control

We aim to reformulate the constrained optimal stopping problem (1.0.1) as a stochastic optimal control problem, by introducing an auxiliary process which can be chosen subject to some constraints, as in [21]. This procedure will give rise to the following two-process stochastic optimal control problem:

$$\begin{aligned} \text{Find} \quad & \sup_{Z,\tau} \mathbb{E}^x[f(B_\tau)] \\ \text{subject to} \quad & Z_t + t \quad \text{is a martingale}, \quad Z_0 = K, \quad Z_t \geq 0 \quad \text{for all} \quad t \leq \tau. \end{aligned} \tag{1.1.1}$$

Here, we proceed to show that problems (1.1.1) and (1.0.1) are in fact equivalent. First we verify that the optimal stopping time in (1.1.1) satisfies the expectation constraint in (1.0.1). Then we show that the set of stopping times over which we optimise is the same in each problem.

Lemma 1.1.1. *Let* $((Z_t)_{t\geq 0}, \tau)$ *be a pair which solves* (1.1.1)*. Then* $\mathbb{E}^x[\tau] \leq K < \infty$*.*

Proof. Suppose that $((Z_t)_{t\geq 0}, \tau)$ solves (1.1.1). Let us introduce the stopping time

$$T_R := \inf\{t \geq 0 : |X_{t\wedge\tau}| \geq R\} \wedge R, \quad \text{for any} \quad R > 0.$$

Then $\tau \wedge T_R$ is a bounded stopping time. Since $Z_t + t$ is a martingale, the optional stopping theorem (see e.g. Theorem 3.2 of [27]) gives us

$$\mathbb{E}[Z_{\tau\wedge T_R} + \tau \wedge T_R] = Z_0.$$

So

$$\begin{aligned}\mathbb{E}[\tau \wedge T_R] &= Z_0 - \mathbb{E}[Z_{\tau\wedge T_R}] \\ &\leq Z_0 = K,\end{aligned} \tag{1.1.2}$$

since $\tau \wedge T_R \leq \tau$ implies that $Z_{\tau\wedge T_R} \geq 0$.

We now wish to take limits as $R \to \infty$. Using the fact that, for $x \in \mathbb{R}^2$, $R \in \mathbb{R}$,

$$|x| \geq R \quad \Rightarrow \quad x_1^2 \geq \frac{R^2}{2} \quad \text{or} \quad x_2^2 \geq \frac{R^2}{2},$$

we see that

$$T_R \geq \min\left\{\inf\left\{t \geq 0 : |B_{t\wedge\tau}| \geq \frac{R}{\sqrt{2}}\right\}, \inf\left\{t \geq 0 : |Z_{t\wedge\tau}| \geq \frac{R}{\sqrt{2}}\right\}, R\right\}.$$

By standard properties of Brownian motion, we know that

$$\inf\left\{t \geq 0 : |B_{t\wedge\tau}| \geq \frac{R}{\sqrt{2}}\right\} \geq \inf\left\{t \geq 0 : |B_t| \geq \frac{R}{\sqrt{2}}\right\} \xrightarrow[R\to\infty]{} \infty, \quad \text{a.s..}$$

Furthermore, $Z_{t\wedge\tau}$ is a super-martingale, stopped at 0, and so

$$\inf\left\{t \geq 0 : |Z_{t\wedge\tau}| \geq \frac{R}{\sqrt{2}}\right\} = \inf\left\{t \geq 0 : Z_{t\wedge\tau} \geq \frac{R}{\sqrt{2}}\right\} \xrightarrow[R\to\infty]{} \infty, \quad \text{a.s..}$$

Hence $T_R \to \infty$ almost surely, as $R \to \infty$. Therefore, taking limits in the inequality (1.1.2), we arrive at

$$\mathbb{E}[\tau] \leq K < \infty.$$

□

Note that, by the martingale representation theorem (see e.g. Theorem 4.3.4 of [24]), $Z_t + t$ being a martingale is equivalent to

$$\mathrm{d}Z_t = \alpha_t \, \mathrm{d}B_t - \mathrm{d}t,$$

for some unique process $\alpha = (\alpha_t)_{t\geq 0}$ adapted to $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, the natural filtration of B.

Lemma 1.1.2. *The constrained optimal stopping problem* (1.0.1) *is equivalent to the stochastic optimal control problem* (1.1.1)*.*

Proof. We first claim that the filtrations generated by the processes B and Z are the same.

As noted above, we can write $\mathrm{d}Z_t = \alpha_t \,\mathrm{d}B_t - \mathrm{d}t$, for some process α adapted to the filtration generated by B. This implies that Z is adapted to the filtration generated by B. Conversely, we can see that B is adapted to the filtration generated by Z, by rearranging the expression. By the definition of generated sigma algebras, we have that the two filtrations must be equal.

This implies that the set of all stopping times with respect to the filtration generated by B is equal to the set of stopping times with respect to the filtration generated jointly by B and any Z. The preceding lemma also tells us that the stopping times in both problems satisfy the constraint $\mathbb{E}[\tau] \leq K$. Therefore, we take the supremum over the same set of stopping times in both problems (1.0.1) and (1.1.1), and so the value function in problem (1.0.1) is equal to the value function in (1.1.1). □

In light of the above lemma, it is the stochastic optimal control problem (1.1.1) which we investigate here.

We will only consider convex reward functions f here, because of the following result, which tells us that, in this case, the solution to the stochastic optimal control problem must allow the process to run for as long as possible. This allows us to fix the stopping time in the problem to be the greatest time allowed by our constraints, thus simplifying the problem. Conversely, if f is concave, the optimal strategy will always be to stop immediately, by the converse of the following lemma. Once we have a good understanding of the solutions for convex f, we will be interested to investigate reward functions which are neither globally convex nor globally concave.

Lemma 1.1.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, B a Brownian motion and τ any stopping time which satisfies the conditions for the optional stopping theorem to apply. Then, for any $\varepsilon > 0$,*

$$\mathbb{E}[f(B_\tau)] \leq \mathbb{E}[f(B_{\tau+\varepsilon})].$$

Proof. Let $\varepsilon > 0$ and suppose that τ satisfies the conditions of the optional stopping theorem for the martingale B. Then

$$\mathbb{E}[B_{\tau+\varepsilon}|\mathcal{F}_\varepsilon] = B_\tau.$$

Therefore, using the convexity of f and the conditional form of Jensen's inequality, we have

$$\begin{aligned}\mathbb{E}[f(B_\tau)] &= \mathbb{E}\left[f\left(\mathbb{E}[B_{\tau+\varepsilon}|\mathcal{F}_\varepsilon]\right)\right] \\ &\leq \mathbb{E}\left[\mathbb{E}[f\left(B_{\tau+\varepsilon}|\mathcal{F}_\varepsilon\right)]\right], \quad \text{by Jensen's inequality,} \\ &= \mathbb{E}[f(B_{\tau+\varepsilon})].\end{aligned}$$

□

From now on, we fix the reward function f to be a convex function. Recall that we have the constraint $Z_t \geq 0$ for all $t \leq \tau$, for any stopping time τ. Therefore, by the above lemma, we have that

$$\sup_{Z,\tau} \mathbb{E}^x[f(B_\tau)] = \sup_{Z} \mathbb{E}^x[f(B_{\tau_0})],$$

where $\tau_0 := \inf\{t \geq 0 : Z_t = 0\}$ is fixed in the expression on the right-hand side.

Also note that Lemma 1.1.3 implies that, for the solution to the constrained optimal stopping problem (1.0.1), we have $\mathbb{E}^x[\tau] = K$. Defining τ as above in the control problem (1.1.1), we see that the condition $\mathbb{E}^x[\tau] = K$ is satisfied, by adapting the proof of Lemma 1.1.1. With this definition of τ, taking limits in equation (1.1.2) gives

$$\mathbb{E}^x[\tau] = Z_0 - \mathbb{E}^x[Z_\tau] = Z_0 = K, \tag{1.1.3}$$

since $Z_\tau = 0$ almost surely.

The stochastic optimal control problem which we wish to solve for convex f is now as follows. Define the set of admissible controls

$$U := \{(\alpha_t)_{t\geq 0} : \alpha \quad \text{adapted to} \quad \mathbb{F}\},$$

and define $X_t = (B_t, Z_t)$, so that

$$\mathrm{d}X_t = \begin{pmatrix} 1 \\ \alpha_t \end{pmatrix} \mathrm{d}B_t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \mathrm{d}t. \tag{1.1.4}$$

Also define the stopping time

$$\tau := \inf\{t \geq 0 : Z_t = 0\}. \tag{1.1.5}$$

We seek the *value function* $v : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ defined as

$$v(x, z) := \sup_{\alpha \in U} \mathbb{E}^{x,z}[f(B_\tau)]. \tag{1.1.6}$$

Notation. The notation $\mathbb{E}^{x,z}$ stands for the expectation under the joint law of B started from $x \in \mathbb{R}$ and Z started from $z \in \mathbb{R}$.

In the remainder of this chapter, we will look at examples where we can explicitly write down the value function, in order to illustrate techniques which we will use in more general cases in future. We will also refer to these examples later to suggest connections to other classes of problems.

1.2 A One-Point Reward Function

We start by taking the reward function f to be a very simple convex function. Let $M \in \mathbb{R}$ and define

$$f(x) = (x - M)_+ \quad \forall x \in \mathbb{R}; \tag{1.2.1}$$

i.e.

$$f(x) = \begin{cases} x - M, & \text{for} \quad x \geq M, \\ 0, & \text{for} \quad x < M. \end{cases}$$

With the stopping time τ defined in (1.1.5), we have $Z_\tau = 0$ almost surely. We conjecture that the optimal strategy will be for the process to run to one of two points, which are spaced equally either side of the midpoint $(M, 0)$ on the z-axis. We expect that, given $X_0 = (x_0, z_0)$, there exist real numbers l, r such that $X_\tau \in \{(l, 0), (r, 0)\}$. Moreover, we should have $l < M < r$, with $M - l = r - M$, so that l and r are related by

$$l = 2M - r, \qquad r = 2M - l,$$

as shown in Figure 1.1.

The value function will then be the expected reward under this optimal strategy. We now deduce the value function which satisfies this requirement.

We see that

$$f(l) = 0, \qquad f(r) = r - M,$$

since $l < M < r$.

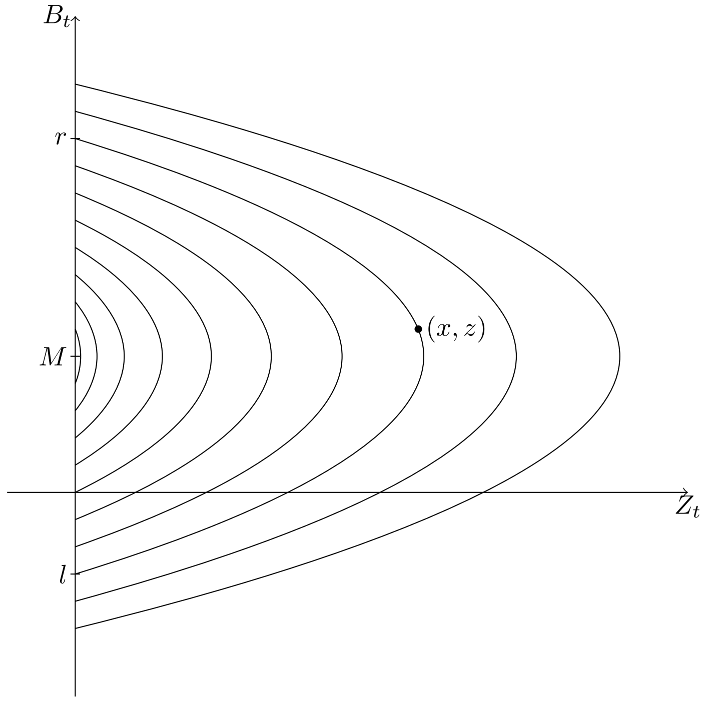

Figure 1.1: The paths that the process would follow under the conjectured optimal strategy for the stochastic optimal control problem (1.1.6) with reward function (1.2.1). We show an example of an initial point (x, z), with its possible terminal points $(l, 0)$, $(r, 0)$.

Let us define the probability $p := \mathbb{P}^{x,z}(B_\tau = r)$ and observe that $\mathbb{P}^{x,z}(B_\tau = l) = 1 - p$. Therefore

$$\mathbb{E}^{x,z}[B_\tau] = pr + (1-p)l = pr + (1-p)(2M - r).$$

Since B_t is a martingale and $B_0 = x$, we can, by introducing a bounded stopping time as in the proof of Lemma 1.1.1, use the optional stopping theorem and a limiting argument to get the condition

$$pr + (1-p)(2M - r) = x. \tag{1.2.2}$$

We also note that, since $\mathbb{E}^x[\tau] < \infty$, Wald's lemma (see e.g. Theorem 2.48 of [22]) implies that $\mathbb{E}^{x,z}[B_\tau^2] = \mathbb{E}^{x,z}[\tau] + x^2$. As noted in (1.1.3) in the previous section, we also have $\mathbb{E}^{x,z}[\tau] = Z_0 = z$. Combining these identities with the formula for $\mathbb{E}^{x,z}[B_\tau^2]$, we get a second condition:

$$pr^2 + (1-p)(2M - r)^2 = z + x^2. \tag{1.2.3}$$

From equation (1.2.2), we can write

$$p = \frac{x + r - 2M}{2(r - M)},$$

and substituting this into equation (1.2.3), we get

$$\frac{x + r - 2M}{2(r - M)} r^2 + \frac{r - x}{2(r - M)} (2M - r)^2 = z + x^2.$$

Rearranging this, we see that

$$(r - M)^2 = (x - M)^2 + z = (l - M)^2,$$

where the final equality is due to the fact that $r - M = M - l$. Then, using the condition $l < M < r$, we can conclude that we must have

$$\begin{aligned} r &= M + \sqrt{z + (x - M)^2} \\ l &= M - \sqrt{z + (x - M)^2}. \end{aligned}$$

We can substitute the above into our expression for p to get

$$p = \frac{x - M + \sqrt{z + (x - M)^2}}{2\sqrt{z + (x - M)^2}}.$$

Given an initial point (x, z), the point l, and therefore r, is fixed. The curves shown in Figure 1.1 are the level curves of l as a function of (x, z).

Under our conjectured optimal strategy, we have

$$\begin{aligned} \mathbb{E}^{x,z}[f(B_\tau)] &= p(r - M)_+ + (1 - p)(l - M)_+ \\ &= p(r - M), \quad \text{since} \quad l < M < r. \end{aligned}$$

Substituting in our expressions for r and p, we get

$$\begin{aligned} \mathbb{E}^{x,z}[f(B_\tau)] &= \frac{x - M + \sqrt{z + (x - M)^2}}{2\sqrt{z + (x - M)^2}} \sqrt{z + (x - M)^2} \\ &= \frac{1}{2}\left(x - M + \sqrt{z + (x - M)^2}\right). \end{aligned}$$

We therefore conjecture the following value function:

$$v(x, z) = \frac{1}{2}\left(x - M + \sqrt{z + (x - M)^2}\right). \tag{1.2.4}$$

In the following section, we aim to prove that v is indeed the value function, using the dynamic programming principle.

1.3 The Dynamic Programming Principle

The dynamic programming principle is a method for deriving a PDE which must be solved in some sense by the value function of a stochastic optimal control problem, as described, for example, in Section 2.2 of [33]. The basic idea is as follows. Suppose that we start at a certain point and, from some intermediate point onwards, we act optimally. Then choosing an optimal strategy is equivalent to finding the optimal strategy up to the intermediate point. We write this down as

$$v(x,z) \geq \mathbb{E}^{x,z}[v(B_\tau, Z_\tau)], \tag{1.3.1}$$

for the value function v and any stopping time τ, with equality for the optimal strategy.

A PDE arises from the above principle by considering the generator of the process. Let us introduce the notation X^α for the process X, defined in (1.1.4), with control $\alpha \in U$, and let $\mathcal{L}^\alpha$ denote the generator of X^α. Recall that the generator is defined by

$$\mathcal{L}^\alpha v(x,z) := \lim_{h \downarrow 0} \frac{\mathbb{E}^{x,z}[v(B_h, Z_h^\alpha)] - v(x,z)}{h},$$

for all functions v such that the limit exists, and for all $x, z \in \mathbb{R}$, as in Section 7.3 of [24].

We therefore see the following approximate relation:

$$\mathbb{E}^{x,z}[v(B_\varepsilon, Z_\varepsilon^\alpha)] = v(x,z) + \mathcal{L}^\alpha v(x,z)\varepsilon + o(\varepsilon),$$

for any $\varepsilon > 0$. Hence, in order to obey the inequality (1.3.1), we require that $\mathcal{L}^\alpha v \leq 0$, with equality for the optimal α. Since the infinitesimal generator is a partial differential operator, we arrive at a PDE which the value function must solve.

In our case, we can use the standard formula for the generator, given in Theorem 7.3.3 of [24], to calculate that

$$\mathcal{L}^\alpha = \frac{\partial}{\partial x_2} + \frac{1}{2}\sum_{i,j}\left[\begin{pmatrix} 1 \\ \alpha_t \end{pmatrix}\begin{pmatrix} 1 \\ \alpha_t \end{pmatrix}^\top\right]\frac{\partial^2}{\partial x_i \partial x_j}.$$

Evaluating

$$\begin{pmatrix} 1 \\ \alpha_t \end{pmatrix}\begin{pmatrix} 1 \\ \alpha_t \end{pmatrix}^\top = \begin{pmatrix} 1 & \alpha_t \\ \alpha_t & \alpha_t^2 \end{pmatrix},$$

we arrive at

$$\mathcal{L}^\alpha = -\frac{\partial}{\partial z} + \frac{1}{2}\frac{\partial^2}{\partial x^2} + \alpha_t \frac{\partial^2}{\partial x \partial z} + \frac{1}{2}\alpha_t^2 \frac{\partial^2}{\partial z^2}. \tag{1.3.2}$$

We now aim to show that our conjectured value function v, defined in (1.2.4), solves a PDE, and to prove formally that any solution to this PDE must solve the stochastic optimal control problem (1.1.6) with reward function (1.2.1).

A connection between the stochastic optimal control problem studied here and the following Monge-Ampère type equation is made in [21]:

$$\frac{1}{2}\det(D^2 u) - \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} = 0.$$

We therefore check that our conjectured value function solves this equation with appropriate conditions. The domain we consider is $\Omega := \mathbb{R} \times (0, \infty)$ with boundary $\partial\Omega := \{(x,z) \in \mathbb{R}^2 : z = 0\}$.

Notation. Throughout, we denote the Hessian of a twice differentiable function u by $D^2 u$.

Remark 1.3.1. The determinant of the Hessian is given by

$$\det(D^2u) = \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial^2 u}{\partial x \partial z}\right)^2.$$

Lemma 1.3.1. *The function v, defined in* (1.2.4)*, is a $C^2(\Omega) \cap C(\overline{\Omega})$ solution to the following PDE problem:*

$$\begin{cases} \frac{1}{2}\det(D^2u) - \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} = 0 & on \quad \Omega \\ \frac{\partial^2 v}{\partial z^2} \leq 0 & on \quad \overline{\Omega} \\ u = f & on \quad \partial\Omega, \end{cases} \tag{1.3.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = (x - M)_+$, for all $x \in \mathbb{R}$.

Proof. We verify our claim by computing the relevant derivatives:

$$\frac{\partial v}{\partial x} = \frac{1}{2}\left(1 + (x-M)(z+(x-M)^2)^{-\frac{1}{2}}\right), \qquad \frac{\partial v}{\partial z} = \frac{1}{4}(z+(x-M)^2)^{-\frac{1}{2}},$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{2}(z+(x-M)^2)^{-\frac{1}{2}} - \frac{1}{2}(x-M)^2(z+(x-M)^2)^{-\frac{3}{2}}, \qquad \frac{\partial^2 v}{\partial z^2} = -\frac{1}{8}(z+(z-M)^2)^{-\frac{3}{2}}.$$

We see immediately that v satisfies the condition $\frac{\partial^2 v}{\partial z^2} \leq 0$ on $\overline{\Omega}$.

Computing the cross derivatives, we see that

$$\begin{aligned} \frac{\partial^2 v}{\partial x \partial z} &= -\frac{1}{4}(x-M)(z+(x-M)^2)^{-\frac{3}{2}} \\ \Rightarrow \left(\frac{\partial^2 v}{\partial x \partial z}\right)^2 &= \frac{1}{16}(x-M)^2(z+(x-M)^2)^{-3}. \end{aligned}$$

Therefore we can calculate the determinant of the Hessian as

$$\begin{aligned} &\det(D^2u) \\ &= \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial^2 u}{\partial x \partial z}\right)^2 \\ &= -\frac{1}{16}(z+(x-M)^2)^{-2} + \frac{1}{16}(x-M)^2(z+(x-M)^2)^{-3} - \frac{1}{16}(x-M)^2(z+(x-M)^2)^{-3} \\ &= -\frac{1}{16}(z+(x-M)^2)^{-2} \\ &= 2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2}. \end{aligned}$$

Hence v satisfies the PDE.
Finally, we check the boundary condition. When $z = 0$, we have

$$\begin{aligned} v(x,0) &= \frac{1}{2}\left(x - M + \sqrt{(x-M)^2}\right) \\ &= \begin{cases} \frac{1}{2}(x - M + (x-M)) = x - M & \text{for} \quad x \geq M \\ \frac{1}{2}(x - M - (x-M)) = 0 & \text{for} \quad x < M. \end{cases} \\ &= (x-M)_+ = f(x), \end{aligned}$$

as required.
Therefore v solves (1.3.3). ☐

We now make rigorous the dynamic programming principle discussed earlier, by showing that the PDE (1.3.3) is the dynamic programming equation, also called the Hamilton-Jacobi-Bellman equation, for the stochastic optimal control problem (1.1.6) with reward function (1.2.1). Our proof of the below theorem follows the arguments in Section 11.2 of [24].

Theorem 1.3.1. *Let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution to the PDE problem* (1.3.3)*:*

$$\begin{cases} \frac{1}{2}\det(D^2u) - \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} = 0 & \text{on } \Omega \\ \frac{\partial^2 v}{\partial z^2} \leq 0 & \text{on } \overline{\Omega} \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = (x - M)_+$, for all $x \in \mathbb{R}$.

Then v is the value function of the stochastic optimal control problem (1.1.1)*; that is*

$$v(x,z) = \sup_{\alpha \in U} \mathbb{E}^{x,z}[f(B_\tau)].$$

Proof. Let $\alpha \in U$ be any admissible control process. In the discussion above, we calculated the generator to be (1.3.2):

$$\mathcal{L}^\alpha = -\frac{\partial}{\partial z} + \frac{1}{2}\frac{\partial^2}{\partial x^2} + \alpha_t \frac{\partial^2}{\partial x \partial z} + \frac{1}{2}\alpha_t^2 \frac{\partial^2}{\partial z^2}.$$

Multiplying through by $2\frac{\partial^2 v}{\partial z^2}$, we get

$$2\frac{\partial^2 v}{\partial z^2}\mathcal{L}^\alpha v = -2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial z^2} + 2\alpha_t \frac{\partial^2 v}{\partial x \partial z}\frac{\partial^2 v}{\partial z^2} + \alpha_t^2\left(\frac{\partial^2 v}{\partial z^2}\right)^2. \tag{1.3.4}$$

Now let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ solve (1.3.3). We then see that, in Ω,

$$\begin{aligned} -2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial z^2} &= -\det(D^2 v) + \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial z^2}, \qquad \text{since } v \text{ solves the PDE}, \\ &= -\left[\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial z^2} - \left(\frac{\partial^2 v}{\partial x \partial z}\right)^2\right] + \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial z^2}, \quad \text{by Remark 1.3.1}, \\ &= \left(\frac{\partial^2 v}{\partial x \partial z}\right)^2. \end{aligned}$$

Substituting this back into (1.3.4), we now have

$$\begin{aligned} 2\frac{\partial^2 v}{\partial z^2}\mathcal{L}^\alpha v &= \left(\frac{\partial^2 v}{\partial x \partial z}\right)^2 + 2\alpha_t \frac{\partial^2 v}{\partial x \partial z}\frac{\partial^2 v}{\partial z^2} + \alpha_t^2\left(\frac{\partial^2 v}{\partial z^2}\right)^2 \\ &= \left(\frac{\partial^2 v}{\partial x \partial z} + \alpha_t \frac{\partial^2 v}{\partial z^2}\right)^2 \\ &\geq 0. \end{aligned}$$

Since $\frac{\partial^2 v}{\partial z^2} \leq 0$, we have that $\mathcal{L}^\alpha v \leq 0$ in Ω.

Moreover, suppose that $\frac{\partial^2 v}{\partial z^2} \neq 0$ and set

$$\alpha^* \equiv -\frac{\partial^2 v}{\partial x \partial z} \bigg/ \frac{\partial^2 v}{\partial z^2}.$$

Then $\mathcal{L}^{\alpha^*}v = 0$ in Ω.

An application of Dynkin's formula (see e.g. Theorem 7.4.1 of [24]) and a limiting argument will now give us that α^* is the optimal control and v is the value function.

Let $\tau := \inf\{t \geq 0 : X_t \in \partial\Omega\} = \inf\{t \geq 0 : Z_t = 0\}$. By Lemma 1.1.1 and the discussion following Lemma 1.1.3, we have that $\mathbb{E}^{x,z}[\tau] < \infty$. However, since v is not compactly supported and Ω is not a compact set, we cannot apply Dynkin's formula directly for the stopping time τ. Therefore, we apply Dynkin to $\tau \wedge T_R$, where $T_R := \inf\{t \geq 0 : |X_{t\wedge\tau}| \geq R\} \wedge R$, as in the proof of Lemma 1.1.1, and use a limiting argument.

We have that $\mathbb{E}^{x,z}[\tau \wedge T_R] \leq \mathbb{E}^{x,z}[\tau] < \infty$ and $\tau \wedge T_R \leq T_R$, with T_R the exit time of a compact domain. Since we also have $v \in C^2(\Omega)$, the conditions to apply Dynkin's formula are satisfied (as set out in Section 7.4 of [24]), and we get

$$\begin{aligned} \mathbb{E}^{x,z}\left[v(B_{\tau\wedge T_R}, Z_{\tau\wedge T_R})\right] &= v(x,z) + \mathbb{E}^{x,z}\left[\int_0^{\tau\wedge T_R} \mathcal{L}^{\alpha} v(B_r, Z_r)\,\mathrm{d}r\right] \\ &\leq v(x,z), \end{aligned} \tag{1.3.5}$$

since $\mathcal{L}^{\alpha}v \leq 0$ in Ω, for all $\alpha \in U$.

We see that $\tau \wedge T_R \to \tau$ almost surely, as $R \to \infty$, since $\mathbb{E}^{x,z}[\tau] < \infty$ implies that $\tau < \infty$ almost surely

By continuity of the value function v, we have $v(B_{\tau\wedge T_R}, Z_{\tau\wedge T_R}) \to v(B_\tau, Z_\tau)$ almost surely, as $R \to \infty$.

Next we wish to apply dominated convergence. We claim that

$$v(x,z) \leq C(x^2 + z + 1), \tag{1.3.6}$$

for some generic constant $C > 0$.

We use the standard inequality

$$\sqrt{a+b} \geq \sqrt{a} + \sqrt{b}, \quad \text{for all} \quad a, b \in \mathbb{R}_+,$$

to see that

$$\sqrt{z + (x-M)^2} \leq \sqrt{z} + (x - M).$$

Then

$$\begin{aligned} v(x,z) &= \frac{1}{2}\left(x - M + \sqrt{z + (x-M)^2}\right) \\ &\leq C(x - M + \sqrt{z}) \leq C(x + \sqrt{z} + 1) \\ &\leq C([x^2+1] + [z+1] + 1), \quad \text{using the inequality } a \leq a^2 + 1, \text{ for any } a \in \mathbb{R}, \text{ twice}, \\ &\leq C(x^2 + z + 1), \quad \text{where } C \text{ is a generic constant throughout.} \end{aligned}$$

Now, by the optional stopping theorem for bounded stopping times,

$$\begin{aligned} \mathbb{E}[B^2_{\tau\wedge T_R}] &= \mathbb{E}[\tau \wedge T_R] < \infty \\ \text{and} \quad \mathbb{E}[Z_{\tau\wedge T_R}] &\leq \mathbb{E}[Z_0] < \infty. \end{aligned}$$

Combining this with the estimate (1.3.6), we get

$$\mathbb{E}^{x,z}\left[v(B_{\tau\wedge T_R}, Z_{\tau\wedge T_R})\right] \leq C\mathbb{E}^{x,z}\left[B^2_{\tau\wedge T_R} + Z_{\tau\wedge T_R} + 1\right] < \infty.$$

The dominated convergence theorem now gives us that, as $R \to \infty$,

$$\mathbb{E}^{x,z}\left[v(B_{\tau\wedge T_R}, Z_{\tau\wedge T_R})\right] \to \mathbb{E}^{x,z}[v(B_\tau, Z_\tau)] = \mathbb{E}^{x,z}[f(B_\tau)].$$

Now, taking limits as $R \to \infty$ in the inequality (1.3.5), we get

$$v(x, z) \geq \mathbb{E}^{x,z}[f(B_\tau)].$$

Finally, if we take $\alpha = \alpha^*$, we have $\mathcal{L}^{\alpha^*} = 0$ in Ω, and so (1.3.5) becomes

$$\mathbb{E}^{x,z}\left[v(B_{\tau\wedge T_R}, Z_{\tau\wedge T_R})\right] = v(x, z).$$

Taking limits, as above, we find that

$$v(x, z) = \mathbb{E}^{x,z}[f(B_\tau)].$$

We have now shown that

$$v(x, z) \geq \mathbb{E}^{x,z}[f(B_\tau)],$$

for any control $\alpha \in U$, with equality for $\alpha = \alpha^*$. Hence v is the value function and α^* is the optimal strategy. □

Combining this theorem and the preceding lemma, we have proved that the function v conjectured in (1.2.4) is indeed the value function for the stochastic optimal control problem (1.1.6) with reward function (1.2.1).

1.4 A Two-Point Reward Function

We now consider a slightly more complicated reward function f, which is the sum of two of the functions of the form we considered in the previous section:

$$f(x) = (x - M_1)_+ + (x - M_2)_+, \quad M_1 < M_2, \quad \text{for all} \quad x \in \mathbb{R}. \tag{1.4.1}$$

Already, we will see that the problem quickly becomes technically more difficult. We use the solution to our first example to conjecture a value function, before introducing the machinery of viscosity solutions in the next section, which will be required to tackle this and more complicated problems.

We make the simplifying assumption that $M_1 = -1$ and $M_2 = 1$ and note that it will be straightforward to generalise this to any M_1, M_2.

We conjecture that the value function will satisfy the PDE problem (1.3.3), as in the previous example, on each of the four regions labelled A, B, C, D in Figure 1.2, but with the boundary condition at $z = 0$ given by f in (1.4.1).

In Lemma 1.3.1 in the previous section, we saw that, for any $M \in \mathbb{R}$, $v(x, z) = \frac{1}{2}(x - M + \sqrt{z + (x - M)^2})$ solves the PDE in (1.3.3). We note that, for any $\alpha, \beta, \gamma, M \in \mathbb{R}$,

$$v(x, z) = \frac{\alpha^2}{2}\left(x - M + \sqrt{z + (x - M)^2}\right) + \beta x + \gamma$$

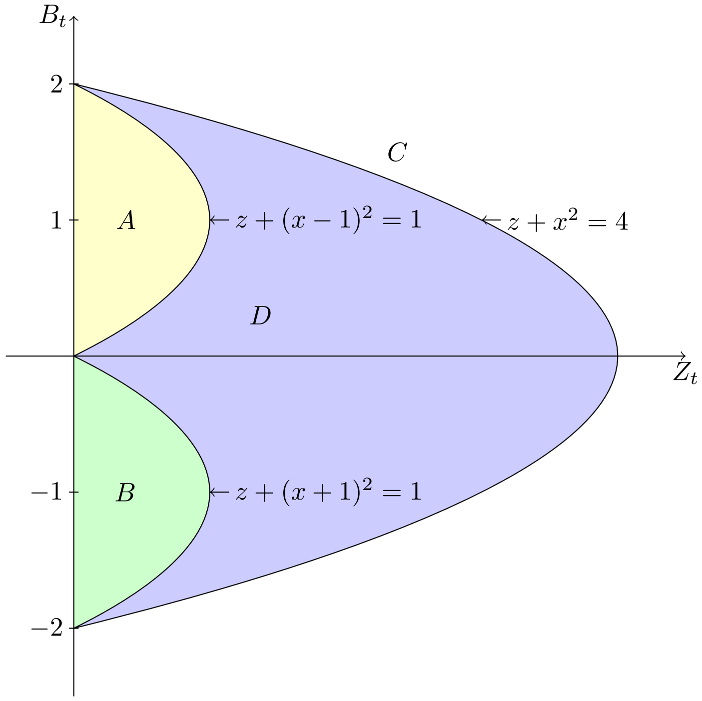

Figure 1.2: The four regions on which the conjectured solution to the optimal stopping problem is defined, with the boundaries between the regions labeled. In regions A, B, C, we expect the optimal trajectories to be of the form shape as those shown in Figure 1.1, while particles started in region D should jump to the nearest boundary between regions, which are themselves optimal trajectories.

is also a solution to this PDE, which maintains the concavity condition $\frac{\partial^2 v}{\partial z^2} \leq 0$, since $\alpha^2 \geq 0$.

In region A, we take $M = 1$. In this region, $x \geq 0$, so the boundary condition is $f(x) = (x-1)_+ + x + 1$, and we must have $\alpha^2 = \beta = \gamma = 1$.

In region B, we take $M = -1$. Here, $x \leq 0$, so the boundary condition is $f(x) = (x+1)_+$, implying that $\alpha^2 = 1$ and $\beta = \gamma = 0$.

In region C, we take $M = 0$. When $z = 0$, we either have $x \leq -2$ or $x \geq 2$, and $f(x) = 2x_+$, in both cases. Therefore $\alpha^2 = 2$ and $\beta = \gamma = 0$.

The solution in region D is not of the same form. Here we seek a function which solves the PDE in such a way that the solution is continuous on the whole domain. We search for a function of the form $v(x, z) = w_1(x, z) + w_2(x, z)$, where w_1, w_2 are linear in z and satisfy

$$w_1(x,z) = \begin{cases} 0, & \text{on} \quad z + (x-1)^2 = 1, \\ x + 2, & \text{on} \quad z + x^2 = 4, \end{cases}$$

$$w_2(x,z) = \begin{cases} \frac{3}{2}x + 1, & \text{on} \quad z + (x-1)^2 = 1, \\ 0, & \text{on} \quad z + x^2 = 4. \end{cases}$$

We see that w_1, w_2 satisfy these conditions if we define them by

$$\begin{aligned} w_1(x,z) &= \frac{z+(x-1)^2-1}{4-x^2+(x-1)^2-1}(x+2), \\ w_2(x,z) &= \frac{z+x^2-4}{1-(x-1)^2+x^2-4}\left(\frac{3}{2}x+1\right). \end{aligned}$$

Summing these two functions results, after some cancellation, in

$$v(x,z) = \frac{1}{4}\left[z+(x+2)^2\right].$$

This also satisfies $v(x,z) = \frac{1}{2}x+1$ on $z+(x+1)^2 = 1$. Therefore v is continuous on the boundaries between D and the regions A, B, C and satisfies the boundary condition at $z = x = 0$.

It is straightforward to check that v satisfies the PDE in D. We can easily calculate the relevant derivatives, as follows:

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{1}{2}(x+2), & \frac{\partial v}{\partial z} &= \frac{1}{4}, \\ \frac{\partial^2 v}{\partial x^2} &= \frac{1}{2}, & \frac{\partial^2 v}{\partial z^2} &= \frac{\partial^2 v}{\partial x \partial z} = 0. \end{aligned}$$

Therefore

$$\frac{1}{2}\det(D^2 v) - \frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial x^2} = \frac{1}{2}\left(\frac{1}{2}\cdot 0 - 0\right) - \frac{1}{4}\cdot 0 = 0,$$

and so the PDE is solved.

Putting all of this together, we conjecture the following value function:

$$v(x,z) = \begin{cases} \frac{1}{2}\left[x-1+\sqrt{z+(x-1)^2}\right]+x+1, & z \in (0, 1-(x-1)^2), x \in (0,2) \\ \frac{1}{2}\left[x+1+\sqrt{z+(x+1)^2}\right], & z \in (0, 1-(x+1)^2), x \in (-2,0) \\ \frac{1}{4}\left[z+(x+2)^2\right], & z \in (\max\{1-(x-1)^2, 1-(x+1)^2\}, 4-x^2), \\ & x \in (-2,2) \\ \frac{1}{2}\left[x+\sqrt{z+x^2}\right], & z > \max 0, 4-x^2, x \in \mathbb{R}. \end{cases} \tag{1.4.2}$$

In the interior of the four regions on which the function is defined, v solves the PDE problem (1.3.3) with boundary function f given by (1.4.1) and so, by a similar argument as in Theorem 1.3.1, v must coincide with the value function for the stochastic optimal control problem.

On some of the boundaries between the regions, however, we can check that v is not continuously differentiable. We calculate, for example, that in region B,

$$\frac{\partial v}{\partial x} = \frac{1}{2}\left[1+(z+(x+1)^2)^{-\frac{1}{2}}(x+1)\right] \to \frac{x}{2}+2,$$

where the limit is taken approaching the boundary between regions B and D, on which $z+(x+1)^2 = 1$. However, in region D, $\frac{\partial v}{\partial x} = \frac{x}{2}$, so v is not C^1 on this boundary. Similar calculations can be made for the boundary between regions C and D. Interestingly, v *is* C^1 on the boundary between regions A and D, but still is not C^2.

Therefore, v cannot be a classical solution to the second order PDE on Ω, and we will have to work harder to show that v is the value function.

In the next section, we introduce a notion of a weak solution known as a viscosity solution. We conjecture that viscosity solutions are the right notion of solution to consider here and that the value function will be the unique viscosity solution to the PDE.

1.5 Viscosity Solutions

Viscosity solutions were introduced by Ishii and Lions in [17] in order to have a good notion of solution for first order Hamilton-Jabobi equations arising in deterministic optimal control problems, as noted in [15]. In [14], Crandall, Ishii and Lions give an overview of viscosity solutions for second order PDEs, such as the Hamilton-Jacobi-Bellman equations which arise in stochastic optimal control problems. In this section, we define viscosity solutions for the type of PDEs in which we are interested, and indicate how these can be applied to the stochastic optimal control problem (1.1.6).

Let $F : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}$ be a differential operator on some domain $\Omega \subset \mathbb{R}^d$. We are interested in the PDE

$$F(x, Du(x), D^2u(x), u(x)) = 0. \tag{1.5.1}$$

As observed in Section II.4 of [15], we require the operator F to be elliptic; i.e.

$$F(x, s, A, V) \leq F(x, s, B, V), \quad \text{for} \quad A - B \text{ non-negative definite.}$$

We introduce the following shorthand notation for positive-definiteness, which we use from now on.

Notation. Let A and B be matrices of the same dimension. We use the notation $A \leq B$ to denote that $A - B$ is a non-positive definite matrix. Similarly, $A < B$ denotes that $A - B$ is negative definite.

We motivate the definition of viscosity solutions as in [14].

Suppose that there exists a classical solution, v, to the PDE and that ϕ is a smooth function which touches the solution from above; i.e. $\phi \geq v$ and $\phi(\bar{x}) = v(\bar{x})$ for some $\bar{x}$. Then $v - \phi$ has a local maximum at $\bar{x}$, which implies that $Dv(\bar{x}) = D\phi(\bar{x})$ and $D^2v(\bar{x}) \leq D^2\phi(\bar{x})$. By ellipticity of F, we have

$$F(\bar{x}, D\phi(\bar{x}), D^2\phi(\bar{x}), v(\bar{x})) \leq F(\bar{x}, Dv(\bar{x}), D^2v(\bar{x}), v(\bar{x})) = 0.$$

Similarly, for a smooth ψ touching v from below, we get $F(\tilde{x}, D\psi(\tilde{x}), D^2\psi(\tilde{x}), v(\tilde{x})) \geq 0$, at the local minimum $\tilde{x}$ of $v - \psi$.

A viscosity solution is a function v which must satisfy the above inequalities for smooth functions touching from above and below, without the requirement that v itself is C^2. This is, therefore, a generalisation of a classical solution for PDEs which are sufficiently well-behaved. We take our definition of a viscosity solution from [15].

Definition 1.5.1 (Viscosity solution)**.** We say that a continuous function $v : \overline{\Omega} \to \mathbb{R}$ is a *viscosity subsolution* of (1.5.1) if, for every smooth $w \in C^\infty(\Omega)$,

$$F(\bar{x}, Dw(\bar{x}), D^2w(\bar{x}), v(\bar{x})) \leq 0,$$

at every point $\bar{x} \in \Omega$ which is a local maximum of $v - w$.

Similarly, a continuous function $v : \overline{\Omega} \to \mathbb{R}$ is a *viscosity supersolution* of (1.5.1) if, for every smooth $w \in C^\infty(\Omega)$,

$$F(\bar{x}, Dw(\bar{x}), D^2w(\bar{x}), v(\bar{x})) \geq 0,$$

at every point $\bar{x} \in \Omega$ which is a local minimum of $v - w$.

A function v which is both a viscosity subsolution and a viscosity supersolution of (1.5.1) is a *viscosity solution.*

We conjecture that the function v defined in (1.1.6) is the unique viscosity solution to the PDE problem (1.3.3), with boundary condition f given in (1.4.1), and we intend to verify this claim.

Using the same method as in the previous example, we can show that v is the value function of the stochastic optimal control problem (1.1.6) with reward function (1.4.1) on the interior of each of the regions A, B, C, D. Using the theory of viscosity solutions described above, we claim that it will be possible to prove that v is the value function on the whole domain. In the following section, we present an alternative approach.

1.6 The Itô-Tanaka Formula

We wish to show that, on each boundary, the expected reward remains the same when following the path of the boundary, but that crossing the boundary will decrease the expected reward. This would complete the proof that v is the desired value function on the domain Ω.

First, we wish to show that $v(B_t, Z_t)$ is a martingale, when (B_t, Z_t) is restricted to lie on one of the boundaries. In order to do this, we will be able to use the one-dimensional form of Itô's formula, since we are looking at a one-dimensional domain and the function v is smooth in this direction. We recall that Itô's formula in one dimension (see e.g. Section 4.1 of [24]) is:

$$\mathrm{d}v(X_t) = \frac{\partial v}{\partial t}(X_t)\,\mathrm{d}t + \frac{\partial v}{\partial x}(X_t)\,\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(X_t)\sigma_t^2\,\mathrm{d}t,$$

for $\mathrm{d}X_t = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t$.

Applying this formula, we expect to be able to show that the expected reward remains constant on each boundary curve.

Next, we wish to show that $v(B_t, Z_t)$ is a supermartingale when moving in any direction which crosses a boundary between two regions. Again, we would like to use Itô's formula to show this, but we have the problem that v is not C^2 at the boundaries. We therefore use the Itô-Tanaka formula (see Chapter VI, §1 of [27]), which allows for functions which are not twice continuously differentiable, by introducing a quantity known as *local time* into the formula. As stated in [27], local time can be thought of as the occupation time of a certain level by a one-dimensional stochastic process.

The Itô-Tanaka formula extends Itô's formula to convex functions and to any function which is the difference of two convex functions. We intend to apply this formula to the example from Section 1.4, and may also make us of this in further examples with different reward functions f.

Chapter 2

Monge-Ampère Equations

Monge-Ampère equations come up in several areas of mathematics, including the theory of optimal transport (see e.g. [10, 34]). A Monge-Ampère equation is defined to be a non-linear second order PDE of the form

$$\det(D^2u) = f(Du, u, x), \tag{2.0.1}$$

as noted in Section 4.1.1 of [34].

Some derivations of stochastic representation results for Monge-Ampère type equations exist in the literature. For example, [32] shows that the solution to a stochastic optimal control problem solves a parabolic Monge-Ampère type equation. We note that the stochastic optimal control problem considered there is similar to the one which we have considered so far in this report, but with additional time-dependence. Similar results have been achieved by Karatzas in [18]. We are not aware, however, of any papers in the literature which give a stochastic representation result for (2.0.1) using the theory of viscosity solutions.

In this chapter we show how stochastic optimal control relates to Monge-Ampère equations and describe how we expect to derive the viscosity solutions to these equations from stochastic optimal control problems. In Section 2.2.1, we provide a preliminary result in this direction.

2.1 A Hamilton-Jacobi-Bellman Equation

The PDE arising in the previous section from the dynamic programming principle is known as a Hamilton-Jacobi-Bellman (HJB) equation (see e.g. Section 11.2 of [24]):

$$\frac{1}{2}\det(D^2u) - \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} = 0.$$

The results of the previous section give us a stochastic representation result for this equation.

It is noted in [21] that this equation is a Monge-Ampère type equation, although it is not quite of the form (2.0.1). Motivated by this observation, we have begun to explore altering our stochastic optimal control problem with the aim of deriving stochastic representation results for Monge-Ampère equations. In the following section, we present one preliminary result which we plan to extend.

2.2 A Stochastic Representation Result

We consider the simplest form of the Monge-Ampère equation in this section:

$$\det(D^2u) = 0.$$

Recall that, so far, we have been considering the stochastic process with generator

$$\mathcal{L}^\alpha = -\frac{\partial}{\partial z} + \frac{1}{2}\frac{\partial^2}{\partial x^2} + \alpha_t \frac{\partial^2}{\partial x \partial z} + \frac{1}{2}\alpha_t^2 \frac{\partial^2}{\partial z^2}.$$

We conjecture that, if we alter this process to remove the $\frac{\partial}{\partial z}$ term from the generator, we will recover the desired form of the Monge-Ampère equation.

Therefore, let us consider the process $X^\alpha = (B, Z)$ defined by

$$\mathrm{d}X_t^\alpha = \begin{pmatrix} 1 \\ \alpha_t \end{pmatrix} \mathrm{d}B_t.$$

The generator of this process is

$$\mathcal{L}^\alpha = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \alpha_t \frac{\partial^2}{\partial x \partial z} + \frac{1}{2}\alpha_t^2 \frac{\partial^2}{\partial z^2}.$$

Returning to the dynamic programming principle described in Section 1.3, we should have $\mathcal{L}^\alpha \leq 0$ for all $\alpha \in U$, with equality for the optimal control.

We see that the generator is maximised for

$$\alpha = \alpha^* = -\frac{\partial^2 u}{\partial x \partial z} \bigg/ \frac{\partial^2 u}{\partial z^2}, \quad \frac{\partial^2 u}{\partial z^2} < 0.$$

Therefore, we expect the value function to satisfy the following PDE:

$$\begin{aligned} 0 = \frac{\partial^2 u}{\partial z^2}\mathcal{L}^{\alpha^*} &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial^2 u}{\partial x \partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x \partial z}\right)^2 \\ &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial z^2} - \frac{1}{2}\left(\frac{\partial^2 u}{\partial x \partial z}\right)^2 \\ &= \frac{1}{2}\det(D^2 u). \end{aligned}$$

We note that the process here has a symmetry in the B and Z directions, since Z is now a martingale. Therefore, we suggest that we should reparametrise the process so that we have a symmetric parametrisation, as follows:

$$\mathrm{d}X_t^\theta = \begin{pmatrix} \mathrm{d}X_t^{1,\theta} \\ \mathrm{d}X_t^{2,\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta_t \\ \sin\theta_t \end{pmatrix} \mathrm{d}B_t,$$

for

$$\theta \in U := \left\{(\theta_t)_{t\geq 0} : \ \theta_t \in [-\pi, \pi), \forall t \geq 0, \ \theta \ \mathbb{F} - \text{adapted}\right\}.$$

In this symmetric problem, we would expect that the condition that the solution is concave in the z direction should be replaced with the symmetric concavity condition that the Hessian matrix D^2v is required to be non-positive definite, which we write $D^2v \leq 0$.

2.2.1 Monge-Ampère on a Compact Domain

We now present an initial stochastic representation result for the Monge-Ampère equation on a compact domain, which we aim to generalise in future work. The method of proof of this theorem is similar to that of Theorem 1.3.1, but is made more straightforward by working on a compact domain.

Theorem 2.2.1. *Let $\Omega \subset \mathbb{R}^2$ be a compact domain. Suppose that there exists a function $v \in C^2(\Omega) \cap C(\overline{\Omega})$ which solves the following Monge-Ampère problem:*

$$\begin{cases} \det(D^2 v) = 0 & \text{on} \quad \Omega \\ D^2 v \leq 0 & \text{on} \quad \Omega \\ v = f & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2.1}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

Then v is the value function of the following stochastic optimal control problem:

$$v(x) = \sup_{\theta \in U} \mathbb{E}^x \left[f(X^\theta_{\tau_{\partial\Omega}}) \right], \quad x \in \mathbb{R}^2, \tag{2.2.2}$$

where

$$\mathrm{d}X^\theta_t = \begin{pmatrix} \cos\theta_t \\ \sin\theta_t \end{pmatrix} \mathrm{d}B_t,$$

and

$$\tau_{\partial\Omega} := \inf\{t \geq 0 : X^\theta_t \in \partial\Omega\}.$$

Proof. We begin by computing the generator of X^θ. Since

$$\begin{pmatrix} \cos\theta_t \\ \sin\theta_t \end{pmatrix} \begin{pmatrix} \cos\theta_t \\ \sin\theta_t \end{pmatrix}^\top = \begin{pmatrix} \cos^2\theta_t & \cos\theta_t \sin\theta_t \\ \cos\theta_t & \sin^2\theta_t \end{pmatrix},$$

the generator is

$$\mathcal{L}^\theta = \frac{1}{2}\cos^2\theta_t \frac{\partial^2}{\partial x_1^2} + \cos\theta_t \sin\theta_t \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{1}{2}\sin^2\theta_t \frac{\partial^2}{\partial x_2^2}.$$

Let v solve the Monge-Ampère problem with the smoothness conditions in the statement of the theorem. Note that the condition that the Hessian is non-positive definite implies in particular that

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}^\top D^2 v \begin{pmatrix} 1 \\ -1 \end{pmatrix} \leq 0.$$

We will make use of this fact, by multiplying the generator by this quantity.

We also use the fact that, since v solves the PDE,

$$\theta_t \frac{\partial^2 v}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} = \det(D^2 v) + \left(\frac{\partial^2 v}{\partial z^2}\right)^2 = \left(\frac{\partial^2 v}{\partial z^2}\right)^2.$$

We now calculate that, for any $\theta \in U$,

$$\begin{aligned}
2\frac{\partial^2 v}{\partial x_1^2}\mathcal{L}^\theta v &= \cos^2\theta_t \frac{\partial^2 v}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} + 2\cos\theta_t \sin\theta_t \frac{\partial^2 v}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_2^2} + \sin_t^2\left(\frac{\partial^2 v}{\partial x_2^2}\right)^2 \\
&= \cos^2\theta_t\left(\frac{\partial^2 v}{\partial z^2}\right)^2 + 2\cos\theta_t\sin\theta_t\frac{\partial^2 v}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_2^2} + \sin_t^2\left(\frac{\partial^2 v}{\partial x_2^2}\right)^2 \\
&= \left(\cos\theta_t \frac{\partial^2 v}{\partial x_1 \partial x_2} + \sin\theta_t \frac{\partial^2 v}{\partial x_2^2}\right)^2.
\end{aligned}$$

Similarly, we can calculate that

$$2\frac{\partial^2 v}{\partial x_2^2}\mathcal{L}^\theta v = \left(\sin\theta_t \frac{\partial^2 v}{\partial x_1 \partial x_2} + \cos\theta_t \frac{\partial^2 v}{\partial x_1^2}\right)^2.$$

Both expressions are clearly non-negative. The first expression is equal to zero, and is therefore minimised, when

$$\tan\theta = -\frac{\partial^2 v}{\partial x_1 \partial x_2}\Bigg/\frac{\partial^2 v}{\partial x_2^2}.$$

Setting the second expression to zero, we find that

$$\tan\theta = -\frac{\partial^2 v}{\partial x_1^2}\Bigg/\frac{\partial^2 v}{\partial x_1 \partial x_2}.$$

Since $\det(D^2 v) = 0$, the above two conditions for θ are, in fact, the same:

$$\begin{aligned}
-\frac{\partial^2 v}{\partial x_1^2}\Bigg/\frac{\partial^2 v}{\partial x_1 \partial x_2} &= -\frac{\partial^2 v}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2}\Bigg/\frac{\partial^2 v}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_2^2} \\
&= -\left(\frac{\partial^2 v}{\partial x_1 \partial x_2}\right)^2\Bigg/\frac{\partial^2 v}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_2^2}, \quad \text{since} \quad \frac{\partial^2 v}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} = \left(\frac{\partial^2 v}{\partial x_1 \partial x_2}\right)^2 \\
&= -\frac{\partial^2 v}{\partial x_1 \partial x_2}\Bigg/\frac{\partial^2 v}{\partial x_2^2}.
\end{aligned}$$

Therefore, we have that

$$2\left(\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2}\right)\mathcal{L}^\theta = \left(\cos\theta_t \frac{\partial^2 v}{\partial x_1 \partial x_2} + \sin\theta_t \frac{\partial^2 v}{\partial x_2^2}\right)^2 + \left(\sin\theta_t \frac{\partial^2 v}{\partial x_1 \partial x_2} + \cos\theta_t \frac{\partial^2 v}{\partial x_1^2}\right)^2 \geq 0,$$

for all $\theta \in U$, with equality when

$$\theta \equiv \theta^* \equiv \arctan\left(-\frac{\partial^2 v}{\partial x_1 \partial x_2}\Bigg/\frac{\partial^2 v}{\partial x_2^2}\right) \in [-\pi, \pi).$$

Since $\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} \leq 0$, we conclude that

$$\mathcal{L}^\theta v \leq 0$$

for all θ, with equality when $\theta \equiv \theta^*$. Now we apply Dynkin's formula, which tells us that

$$\begin{aligned}
\mathbb{E}^x\left[v(X^\theta_{\tau_{\partial\Omega}})\right] &= v(x) + \mathbb{E}^x\left[\int_0^{\tau_{\partial\Omega}} \mathcal{L}^\theta v(X_t^\theta)\,\mathrm{d}t\right] \\
&\leq v(x),
\end{aligned} \tag{2.2.3}$$

since $\mathcal{L}^\theta v(x) \leq 0$ for all $\theta \in U$ and all $x \in \mathbb{R}^2$.

Applying the boundary condition gives us that $v(X^\theta_{\tau_{\partial\Omega}}) = f(X^\theta_{\tau_{\partial\Omega}})$ almost surely, by continuity of f, so we have

$$v(x) \geq \mathbb{E}^x \left[f(X^\theta_{\tau_{\partial\Omega}}) \right], \quad \text{for all} \quad \theta \in U, x \in \mathbb{R}^2.$$

Moreover, for $\theta \equiv \theta^*$, we have equality in the expression (2.2.3), so

$$v(x) = \mathbb{E}^x \left[f(X^{\theta^*}_{\tau_{\partial\Omega}}) \right].$$

Hence v is the value function for the stochastic control problem (2.2.2), with optimal control θ^*. $\square$

Recalling the example in Section 1.4, we expect that, for most choices of f, the value function (2.2.2) will not be C^2 and therefore cannot solve the Monge-Ampère problem (2.2.1) in the classical sense.

We now verify that viscosity solutions, as defined in Definition 1.5.1, will be the right notion of solution to consider in this case.

Remark 2.2.1. Suppose that there exists a classical solution $v \in C^2(\Omega) \cap C(\partial\Omega)$ to the Monge-Ampère problem (2.2.1) in the above theorem. Then v is also a viscosity solution to the PDE.

We will appeal to the following lemma to prove this.

Lemma 2.2.1. *Let A be a 2×2 matrix which is either positive semi-definite or negative semi-definite. Then* $\det A \geq 0$.

Proof. First suppose that $A \geq 0$ and let v be an eigenvector of A with corresponding eigenvalue λ. Then

$$0 \leq v^\top A v = v^\top \lambda v = \lambda \|v\|^2.$$

So $\lambda \geq 0$. Since A has two eigenvalues, λ_1, λ_2, which are both non-negative, we have

$$\det A = \lambda_1 \lambda_2 \geq 0.$$

Now suppose $A \leq 0$. Similarly to above,

$$0 \geq \lambda \|v\|^2,$$

for any eigenpair (λ, v). So $\lambda_1, \lambda_2 \leq 0$, and hence

$$\det A = \lambda_1 \lambda_2 \geq 0.$$

$\square$

Proof of remark. Let $v \in C^2(\Omega) \cap C(\partial\Omega)$ be a classical solution to (2.2.1). First, we show that v is a viscosity subsolution. Let $w \in C^\infty(\Omega)$ and $x \in \arg\max(v - w)$. We need to show that $\det(D^2 w)(x) \leq 0$.

Since x is a local maximum of $v - w$, $D^2(v - w)(x) \leq 0$. Therefore, by Lemma 2.2.1, $\det D^2(v - w)(x) \geq 0$. Since v is a classical solution to the PDE,

$$\det(D^2 w)(x) \leq \det(D^2 v)(x) = 0.$$

Hence v is a viscosity subsolution of the PDE. To show that v is also a supersolution, now let $y \in \arg\min(v - w)$. We now need to show that $\det(D^2 w)(y) \geq 0$.

Since y is a local minimum of $v - w$, $D^2(v - w)(y) \geq 0$. Also, by our assumptions on v, we know that $D^2 v \leq 0$. Therefore,

$$(D^2 w)(y) \leq (D^2 v)(y) \leq 0.$$

Applying Lemma 2.2.1 again, we have $\det(D^2 w)(y) \geq 0$, as required. Hence we have shown that v is a viscosity solution. □

We will aim to prove that, in general, the value function of the stochastic optimal control problem (2.2.2) is a viscosity solution to the Monge-Ampère problem. We also expect that we will be able to prove uniqueness of this viscosity solution using a *comparison principle*, as explained in [15] and proved for the problem (1.3.3) by Miller in [21].

Conjecture 2.2.1. *Let $\Omega \subset \mathbb{R}^2$ be a compact domain and let v be the value function of the following stochastic optimal control problem:*

$$v(x) = \sup_{\theta \in U} \mathbb{E}^x \left[f(X^\theta_{\tau_{\partial\Omega}}) \right], \quad x \in \mathbb{R}^2,$$

where

$$\mathrm{d}X^\theta_t = \begin{pmatrix} \cos\theta_t \\ \sin\theta_t \end{pmatrix} \mathrm{d}B_t,$$

and

$$\tau_{\partial\Omega} := \inf\{t \geq 0 : X^\theta_t \in \partial\Omega\}.$$

Then v is the unique viscosity solution to the following Monge-Ampère problem:

$$\begin{cases} \det(D^2 v) = 0 & on \quad \Omega \\ D^2 v \leq 0 & on \quad \Omega \\ v = f & on \quad \partial\Omega, \end{cases}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

We have made the assumption throughout this section that Ω is compact, so that Dynkin's formula can be applied immediately in the above proof. However, we may wish to consider domains which are not compact. This brings us to escape problems, which we consider in the next section.

2.2.2 Escape Problems

We may be interested to extend any stochastic representation results which we derive on compact domains to non-compact domains, such as the half-plane (as in the example of Section 1.1) or a strip ($\{x \in \mathbb{R} : m_1 \leq x_2 \leq m_2\}$). We expect that, on these domains, with boundary conditions defined appropriately, stochastic representation results for the Monge-Ampère equation will still hold.

We conjecture that there is a connection here to another class of problems, which we call *escape problems*. This type of problem involves finding the probability that, given a stochastic process started in a certain region of a domain, the process will leave this region before hitting the boundary of the domain.

For example, considering the domain in Figure 2.1, we may be interested in the probability that a process started in region A escapes into region B before hitting the boundary of the domain in region A.

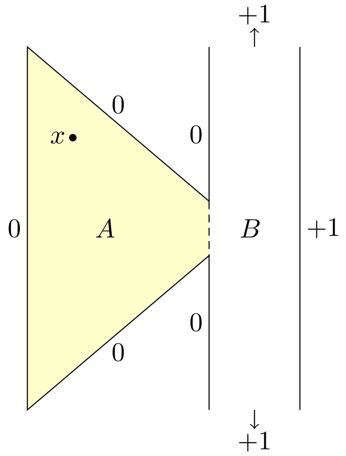

Figure 2.1: The domain of an escape problem, with rewards assigned to each boundary, so that the probability of a stochastic process started from some point x escaping from the shaded region A is equal to the value function of a stochastic optimal control problem evaluated at x.

By carefully defining the reward function, we can write such an escape problem as a stochastic optimal control problem. In our example, we may choose to assign a reward of $+1$ for hitting the boundary of the domain in region B and likewise for the process tending to $\pm\infty$, but assign zero reward for hitting the boundary of the domain in region A, as shown in Figure 2.1. Then we expect the value function on A to be equal to the probability of escape from region A.

Because of this relationship between escape problems and stochastic optimal control problems, we suggest that, with appropriately defined reward functions, optimal strategies for escape problems may give viscosity solutions to a Monge-Ampère equation.

2.3 Connections to Optimal Transport

One important application of Monge-Ampère equations is in optimal transport, as described, for example, in [10] and in Chapter 4 of [34].

The Monge-Kantorovich problem of optimal transport is defined in [34] as follows:

Let μ, ν be probability measures on $\mathbb{R}$ such that $\int_{\mathbb{R}} x \, d\mu < \infty$, $\int_{\mathbb{R}} x \, d\nu < \infty$, and define a *cost function* $c : \mathbb{R}^2 \to \mathbb{R}$. Then the optimal transport problem is to find

$$C(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^2} c(x, y) \, d\pi(x, y),$$

where $\Pi(\mu,\nu) := \{\pi$ a probability measure : marginals of π are μ, $\nu.\}$ is the set of all *transport plans* or *couplings*.

Let us consider the quadratic cost function $c(x,y) := |x-y|^2$. Then Theorem 2.12 of [34] tells us that, provided μ and ν have finite second moments, there exists a unique optimal transport plan π supported by the graph of a function u, which is the gradient of a convex function.

Moreover, suppose that each measure is absolutely continuous with respect to Lebesgue measure, having density f and g, respectively. Then it is stated in [34] that u satisfies the following form of the Monge-Ampère equation:

$$\det(D^2u(x)) = \frac{f(x)}{g(\nabla u(x))}. \tag{2.3.1}$$

The interpretation of the solution as the gradient of a convex function and the connection to the Monge-Ampère equation were first observed by Brenier in [7] (translated in [8]), and regularity of this PDE was shown by Caffarelli in [9]. We intend to investigate stochastic representation results, in terms of stochastic optimal control problems, for this form of the Monge-Ampère equation.

One approach which we could try is to change the reward function so that it depends on the whole path that the process takes, as well as its final position, as in the general statements of stochastic optimal control problems in Section 2.1 of [33] and Chapter 11 of [24]. For example, we could look at the problem of finding

$$v(x) := \sup_{\theta\in U}\mathbb{E}^x\left[f(X^\theta_{\tau_{\partial\Omega}}) + \int_0^{\tau_{\partial\Omega}} g(X^\theta_t)\,\mathrm{d}t\right],$$

or

$$v(x) := \sup_{\theta\in U}\mathbb{E}^x\left[f(X^\theta_{\tau_{\partial\Omega}})e^{-\int_0^{\tau_{\partial\Omega}} h(X^\theta_t)\,\mathrm{d}t}\right],$$

for some appropriate functions $g, h : \mathbb{R}\to\mathbb{R}$.

It is unclear whether changing the reward function in such a way will yield the Monge-Ampère equation (2.3.1), but we are nonetheless interested to find an interpretation of these value functions in terms of PDEs, and we will also consider other ways of adapting the stochastic optimal control problem to connect it to the desired form of the Monge-Ampère equation.

Chapter 3

Martingale Optimal Transport and Skorokhod Embeddings

Martingale optimal transport problems are characterised as classical Monge-Kantorovic problems, with an extra constraint on the admissible transport plans. In this chapter we will define this class of problems and present some first properties. Connections have been made between (martingale) optimal transport and Skorokhod embedding problems in [4, 5], and we also introduce Skorokhod embedding in this chapter. We suggest possible links between both classes of problems and constrained optimal stopping problems, which we propose to investigate further.

3.1 Optimal Transport

Having introduced the classical Monge-Kantorovich optimal transport problem in the previous section, we now define martingale optimal transport and describe monotonicity principles for both the classical and martingale problems, following Beiglböck and Juillet's presentation in [6]. This will lead us to propose a connection to constrained optimal stopping.

Let μ, ν be probability measures on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} x \, \mathrm{d}\mu < \infty$, $\int_{\mathbb{R}} x \, \mathrm{d}\nu < \infty$, and let $c : \mathbb{R}^2 \to \mathbb{R}$ be a cost function, which we now assume satisfies the following integrability condition, as required in [6]:

$$c(x, y) \geq a(x) + b(y), \quad \text{for all} \quad x, y \in \mathbb{R}, \tag{3.1.1}$$

for some $a \in \mathcal{L}^1(\mu)$, $b \in \mathcal{L}^1(\nu)$.

The optimal transport problem is to find

$$C(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^2} c(x, y) \, \mathrm{d}\pi(x, y),$$

where the infimum is taken over the set of all transport plans, as defined in Section 2.3.

We next impose a further constraint to arrive at martingale optimal transport.

3.1.1 Martingale Optimal Transport

To define a martingale optimal transport problem, we introduce the following subset of $\Pi(\mu,\nu)$:

$$\Pi_M(\mu,\nu) := \{\pi \in \Pi(\mu,\nu) : \pi = \mathrm{Law}(X,Y), \mathbb{E}(Y|X) = X\},$$

as in [6]. Then the martingale optimal transport problem is to find

$$C_M(\mu,\nu) := \inf_{\pi\in\Pi_M(\mu,\nu)} \int_{\mathbb{R}^2} c(x,y)\,\mathrm{d}\pi(x,y). \tag{3.1.2}$$

The name martingale optimal transport is used here, since the additional condition imposed is equivalent to requiring that the two-step process $(X_n)_{n=1,2}$ is a martingale, where $\mathrm{Law}(X_1) = \mu$, $\mathrm{Law}(X_2) = \nu$.

Remark 3.1.1. Note that the set of transport plans, $\Pi(\mu,\nu)$ is non-empty, since the product measure, $\mu \otimes \nu$ is always a member of this set.

However, the set of martingale transport plans, $\Pi_M(\mu,\nu)$ may be empty. In fact, it is noted in [6] that this set is non-empty if and only if the measures μ and ν are in *convex order* $\mu \leq \nu$, as defined below.

Definition 3.1.1 (Convex order)**.** Let μ,ν be probability measures on $\mathbb{R}$. We say that $\mu \leq \nu$ in *convex order* if

$$\int_{\mathbb{R}} \phi\,\mathrm{d}\mu \leq \int_{\mathbb{R}} \phi\,\mathrm{d}\nu,$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}_+$.

3.1.2 Monotonicity

In the classical theory of optimal transport, there is a well-known monotonicity principle. Recall from Section 2.3 that, for the case of quadratic cost, there exists unique optimal coupling, which is supported by the graph of the gradient of a convex function. Since the gradient of a convex function is a monotonically increasing function, the optimal coupling is the monotone coupling, as observed in [6]; i.e. if x_1 is mapped to y_1 and x_2 mapped to y_2, then $x_1 < x_2$ implies $y_1 < y_2$.

Theorem 1.1 of [6] states a more general monotonicity result that, for any cost function of the form $c(x,y) = h(x-y)$, for some strictly convex function $h : \mathbb{R} \to \mathbb{R}$. Namely, any optimal coupling is monotone, where we say that $\pi \in \Pi(\mu,\nu)$ is *monotone* if there exists a set $\Gamma \subset \mathbb{R}^2$ with $\pi(\Gamma) = 1$ such that $x_1 < x_2$ implies $y_1 < y_2$ for all pairs $(x_1,y_1),(x_2,y_2) \in \Gamma$.

We now describe a similar monotonicity principle for martingale optimal transport, as proved by Beiglböck and Juillet in [6].

Definition 3.1.2 (Monotonicity)**.** We say that a measure $\pi \in \Pi_M(\mu,\nu)$ is *(left-)monotone* if there exists a Borel-measurable set, $\Gamma \subset \mathbb{R}^2$, with $\pi(\Gamma) = 1$, such that, for all pairs $(x,y^-),(x,y^+),(x',y') \in \Gamma$,

$$x < x' \quad \Rightarrow \quad y' \in (-\infty, y^-] \cup [y^+, +\infty).$$

We call Γ the *monotonicity set* of π.

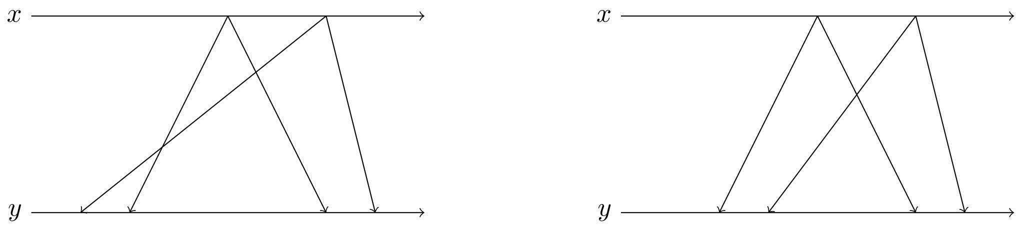

Figure 3.1: These diagrams, adapted from Figure 1 in [6], each show mass being transported from two points at x to four points at y. On the left, we show a mapping which is allowed to belong to the monotonicity set of a left-monotone coupling. On the right, we show an example of a mapping which is forbidden.

We illustrate a mapping which satisfies this definition and one which does not in Figure 3.1.

Using the above definition of monotonicity, we have the following result from [6].

Theorem 3.1.1. *Let μ, ν be probability measures on $\mathbb{R}$, with $\mu \leq \nu$ in convex order, and suppose that the cost function c satisfies the integrability condition* (3.1.1)*. Then there exists a unique left-monotone measure $\pi_{lc} \in \Pi_M(\mu, \nu)$, which we call the* left-curtain coupling.

We recall the initial example of a constrained optimal stopping problem which we considered in Section 1.2. The trajectories of the optimal solution to this problem, as illustrated in Figure 1.1, appear to obey a monotonicity principle similar to that which the left-curtain coupling satisfies in a martingale optimal transport problem. Having also observed in the previous chapter that Monge-Ampère equations are related to both optimal transport and constrained optimal stopping, we conjecture that there is a relationship between the constrained optimal stopping problem which we have considered here and martingale optimal transport.

Inspired by [4] and [5], in which the authors have exposed connections between optimal transport and Skorokhod embedding, we now turn our attention to Skorokhod embedding problems.

3.2 Skorokhod Embedding Problems

In this section, we define what is meant by a Skorokhod embedding problem, introduce some solutions from the literature and describe this problem's connection to optimal transport and optimal stopping. We take our definition of the Skorokhod embedding problem from the review of Obłój [23].

Let μ be a probability measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} |x| \,\mathrm{d}\mu(x) < \infty$ and $\int_{\mathbb{R}} x \,\mathrm{d}\mu(x) = 0$, and let B be standard Brownian motion on $\mathbb{R}$. The problem is then to find a stopping time τ such that

$$B_\tau \sim \mu \quad \text{and} \quad (B_{t\wedge\tau})_{t\geq 0} \quad \text{is a uniformly integrable martingale.} \tag{3.2.1}$$

Note that the condition of uniform integrability can be replaced by the condition that $\mathbb{E}[\tau] < \infty$, as in the definition in [4]. This implies that $\int_{\mathbb{R}} x^2 \,\mathrm{d}\mu(x) = \mathbb{E}[\tau] < \infty$.

The embedding problem was first posed by Skorokhod in 1961 [30] (translated in [31]), where he also proposed the first solution. It is noted in [23] that, since Skorokhod's initial work, many different solutions have been found using methodology from almost every area of the theory of stochastic processes, each method having their own nice properties. For example, Chacon and Walsh used tools from potential theory in [11] as a simple method to solve the Skorokhod embedding problem. The Azéma-Yor stopping time suggested in [2] is more explicit and has the property (noted in [23]) that it maximises the law of the maximum of the process. This is actually a special case of the Chacon-Walsh solution, as shown in [12]. We will focus here on Root's construction, first given in [28], as it will allow us to demonstrate connections with optimal transport and optimal stopping. We describe this construction as in [23].

Theorem 3.2.1 (Root's construction)**.** *Let μ be a probability measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} x^2 \,\mathrm{d}\mu(x) < \infty$ and $\int_{\mathbb{R}} x \,\mathrm{d}\mu(x) = 0$. Then there exists a* barrier $\mathcal{R} \subset \mathbb{R}_+ \times \mathbb{R}$ *such that the stopping time*

$$\tau := \inf\{t \geq 0 : (t, B_t) \in \mathcal{R}\}$$

solves the embedding problem (3.2.1)*. We say that $\mathcal{R}$ is a* barrier *if it is a closed set satisfying the condition that $(t, x) \in \mathcal{R}$ implies $(s, x) \in \mathcal{R}$, for any $s > t$, and furthermore, $(+\infty, x) \in \mathcal{R}$, for all $x \in [-\infty, +\infty]$, and $(t, \pm\infty) \in \mathcal{R}$ for all $t \in [0, +\infty]$.*

3.2.1 Connections to Optimal Stopping

Root's solution can be characterised by a corresponding optimal stopping problem, as described in [13]. Define the *potential* of any measure μ such that $\int_{\mathbb{R}} |x| \,\mathrm{d}\mu(x) < \infty$ and $\int_{\mathbb{R}} x \,\mathrm{d}\mu(x) = 0$ as

$$U^\mu(x) := -\int_{\mathbb{R}} |y - x| \,\mathrm{d}\mu(y).$$

Then the optimal stopping problem of finding

$$u(x, t) := \sup_{\tau \leq t} \mathbb{E}^x \left[U^\lambda(X_\tau) \mathbb{1}_{\{\tau = t\}} + U^\mu(X_\tau) \mathbb{1}_{\{\tau < t\}} \right]$$

yields the stopping region

$$\mathcal{R} := \left\{ (t, x) \in [0, \infty] \times [-\infty, \infty] : \; u(x, t) = U^\lambda(x) \right\}.$$

The main result of [13] then shows that this region $\mathcal{R}$ is the barrier corresponding to the Root solution of the Skorokhod embedding problem (3.2.1). In Figure 3.2, we see an example of such a barrier $\mathcal{R}$. The horizontal lines inside the barrier indicate that, in the optimal stopping problem, if it is optimal to stop at a certain level at one time, then we should also stop if we hit that same level at any later time.

We note that this optimal stopping problem includes a time variable, which we have not considered so far in this report. We intend to investigate this type of problem and consider its relationship to problems of optimal stopping with constraints.

3.2.2 Connections to Optimal Transport

Another property of Root's solution is noted in [4]. Namely, Root's solution is optimal in the sense that it minimises the quantity $\mathbb{E}[\tau^2]$ over all possible embeddings. This leads us to define the optimal Skorokhod embedding problem, as in [4]. Define the set of stopped paths

$$S := \{(f, s) : f \in C([0, s], \mathbb{R}), f(0) = 0\}.$$

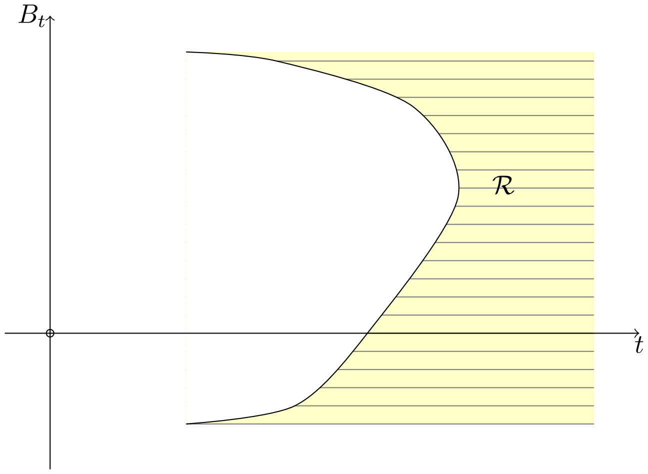

Figure 3.2: An example of the barrier in Root's solution to the Skorokhod embedding problem, adapted from Figure 1 of [4]. The Brownian motion starts from the origin, and the shaded region $\mathcal{R}$ is the barrier. The grey horizontal lines indicate that, if the Brownian motion takes some value x at time t_1 and is inside the barrier, and if there is a later time $t_2 > t_1$ when it takes the same value x, then it will also be inside the barrier at that later time.

and consider a functional $\gamma : S \to \mathbb{R}$. Then the optimal Skorokhod embedding problem is to find

$$P_\gamma(\mu) := \inf \left\{ \mathbb{E}[\gamma((B_t)_{t\le\tau}, \tau)] : \tau \text{ solves } (3.2.1) \right\}. \tag{3.2.2}$$

With this definition, we see that Root's construction solves the optimisation problem (3.2.2) for the functional defined by $\gamma(f, s) = s^2$.

In [4], Beiglböck, Cox and Huesmann apply ideas from both the theory of optimal transport and stochastic analysis to the optimal Skorokhod embedding problem in order to construct solutions to (3.2.2) for different functionals γ. This gives rise to a wide class of solutions to the Skorokhod embedding problem (3.2.1).

Furthermore, in [5], the left-curtain coupling from martingale optimal transport (defined in Theorem 3.1.1) is interpreted in terms of a Skorokhod embedding problem. In particular, the authors show in this paper that the left-curtain coupling gives rise to a barrier, as defined in Theorem 3.2.1, from which a solution to a Skorokhod embedding problem can be constructed.

Combining this with the observations made in the previous sections of this report, we expect a connection between the optimal Skorokhod embedding problem (3.2.2), the martingale optimal transport problem (3.1.2), the Monge-Ampère equation (2.0.1) and the constrained optimal stopping problem (1.0.1), and we intend to undertake future work to investigate this conjectured relationship.

Outline of Future Work

We conclude by summarising the research questions which we plan to investigate in the forthcoming PhD thesis, as described in this report. Our initial aims are as follows:

- To complete the example of Section 1.4 using the theory of viscosity solutions described in Section 1.5, and to investigate other reward functions f for which we can write down an explicit value function for the constrained optimal stopping problem (1.0.1).
- To prove the stochastic representation result for the Monge-Ampère equation in Conjecture 2.2.1, to investigate the relationship between Monge-Ampère equations and escape problems, as defined in 2.2.2, and to seek a stochastic representation result for the version of the Monge-Ampère equation (2.3.1) which arises in optimal transport.
- To study in more depth both martingale optimal transport, as described in Section 3.1.1, and Skorokhod embedding problems, defined in Section 3.2, and to investigate connections between these and problems of stochastic optimal control and constrained optimal stoppping.

This outline represents our views of interesting directions of research to pursue at the present time, and we expect that several further questions of interest will arise during the course of the PhD, as we explore in more depth the problems discussed in this report and the connections between them.

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